Generalized Relative Entropies, Entanglement Monotones and a family of quantum protocols

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Quantum Relative Entropy

- A fundamental quantity in Quantum Mechanics & Quantum Information Theory is the Quantum Relative Entropy

  of ρ w.r.t. σ, $\rho \geq 0, \ \mathrm{Tr}\, \rho = 1$ $\quad \sigma \geq 0$:

  (state / density matrix)

$$S(\rho \| \sigma) := \mathrm{Tr}\,(\rho \log \rho) - \mathrm{Tr}\,(\rho \log \sigma)$$

$\log \equiv \log_2$

well-defined if $\quad \mathrm{supp}\, \rho \subseteq \mathrm{supp}\, \sigma$

- It acts as a parent quantity for the *von Neumann entropy*:

$$S(\rho) := -\mathrm{Tr}\,(\rho \log \rho) = -S(\rho \| I) \qquad (\sigma = I)$$

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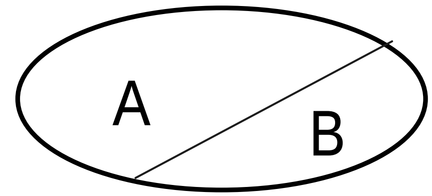
- It also acts as a parent quantity for other entropies:

e.g. for a bipartite state ρ_{AB} :



- *Conditional entropy*

$$S(A|B)_\rho := S(\rho_{AB}) - S(\rho_B) = -S(\rho_{AB} \,||\, I_A \otimes \rho_B)$$

- *Mutual information*

$$\rho_B = \mathrm{Tr}_A\, \rho_{AB}$$

$$I(A:B)_\rho := S(\rho_A) + S(\rho_B) - S(\rho_{AB}) = S(\rho_{AB} \,||\, \rho_A \otimes \rho_B)$$

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Some Properties of $S(\rho \| \sigma)$

"distance"

$$S(\rho \| \sigma) \geq 0 \qquad \rho, \sigma \text{ states}$$
$$= 0 \text{ if \& only if } \rho = \sigma$$

- Joint convexity:

For two mixtures of states $\rho = \sum_{i=1}^{n} p_i \rho_i$ & $\sigma = \sum_{i=1}^{n} p_i \sigma_i$

$$S(\rho \| \sigma) \leq \sum_k p_k S(\rho_k \| \sigma_k)$$

- Invariance under

joint unitaries

$$S(U \rho U^* \| U \sigma U^*) = S(\rho \| \sigma)$$

- Monotonicity of Quantum Relative Entropy under a completely positive trace-preserving (CPTP) map Λ:

v.powerful!

$$S(\Lambda(\rho) \| \Lambda(\sigma)) \leq S(\rho \| \sigma) \qquad \text{..........(1)}$$

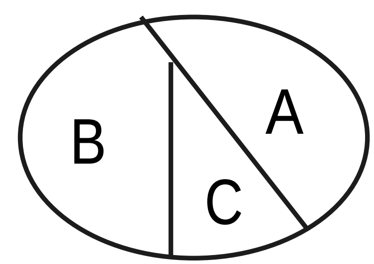
- Many properties of other entropies can be proved using (1)

e.g. *Strong subadditivity of the von Neumann entropy*

Lieb & Ruskai '73

$$S(\rho_{\mathrm{ABC}}) + S(\rho_{\mathrm{B}}) \leq S(\rho_{\mathrm{AB}}) + S(\rho_{\mathrm{BC}})$$

$$S(A|BC)_\rho \leq S(A|B)_\rho$$



ρ_{ABC}

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Outline

- *Define 2 generalized relative entropy quantities*
- *Discuss their properties and operational significance*
- *Define 2 entanglement monotones*
- *Discuss their operational significance*
- *Consider a family tree of quantum protocols*

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Two new relative entropies

- *Definition 1 :* The max- relative entropy of a state ρ & a positive operator σ is

$$S_{\max}(\rho \| \sigma) := \log \left(\min \{ \lambda : \rho \leq \lambda \sigma \} \right)$$

$$(\lambda\sigma - \rho) \geq 0$$

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- *Definition 2:* The min- relative entropy of a state ρ & a positive operator σ is

$$S_{\min}(\rho \| \sigma) := -\log \operatorname{Tr}(\pi_\rho \sigma)$$

where π_ρ denotes the projector onto the support of ρ $(\operatorname{supp} \rho)$

$$\operatorname{supp} \rho \cap \operatorname{supp} \sigma \neq \emptyset$$

- *Remark:* The min- relative entropy

$$S_{\min}(\rho\|\sigma) := -\log \operatorname{Tr} \pi_\rho \sigma$$

is the *quantum relative Renyi entropy* of order 0 :

$$S_{\min}(\rho\|\sigma) = S_0(\rho\|\sigma) = \lim_{\alpha\to 0^+} S_\alpha(\rho\|\sigma)$$

where

$$S_\alpha(\rho\|\sigma) := \frac{1}{\alpha-1}\log \operatorname{Tr}(\rho^\alpha\sigma^{1-\alpha})$$

quantum relative Renyi entropy of order α

$(\alpha \neq 1)$

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$$S_{\max}(\rho\|\sigma) \geq S_{\min}(\rho\|\sigma)$$

- *Proof:*

$$S_{\max}(\rho\|\sigma) := \log\left(\min\{\lambda : \rho \leq \lambda\sigma\}\right) = \log \lambda_0$$

$$\rho \leq \lambda_0\sigma, \qquad (\lambda_0\sigma - \rho) \geq 0$$

$$\mathrm{Tr}\,[\pi_\rho(\lambda_0\sigma - \rho)] \geq 0 \qquad \because \pi_\rho \geq 0$$

$$\lambda_0 \mathrm{Tr}\,[\pi_\rho\sigma] \geq \mathrm{Tr}\,[\pi_\rho\rho] = 1$$

$$\log\lambda_0 + \log\,[\mathrm{Tr}(\pi_\rho\sigma)] \geq 0$$

$$\log\lambda_0 \geq -\log\,[\mathrm{Tr}(\pi_\rho\sigma)]$$

$$S_{\max}(\rho\|\sigma) \geq S_{\min}(\rho\|\sigma)$$

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Operational significance of $S_{\min}(\rho \| \sigma)$

- *State Discrimination:* Bob receives a state

ρ or σ

- He does a measurement to infer which state it is

POVM $\Pi\ [\rho]$ & $(I-\Pi)\ [\sigma]$ $0 \le \Pi \le I$

Possible errors	*inference*	*actual state*
Type I	σ	ρ
Type II	ρ	σ

hypothesis testing

- *Error probabilities*

$\alpha = \mathrm{Tr}((I-\Pi)\rho)$ *Type I*

$\beta = \mathrm{Tr}(\Pi\sigma)$ *Type II*

- *Suppose* $\Pi = \pi_\rho$ (POVM element)

Prob(Type I error)

$$\alpha = \mathrm{Tr}((I - \Pi)\rho)$$
$$= 0$$

Prob(Type II error)

$$\beta = \mathrm{Tr}(\Pi\sigma)$$
$$= \mathrm{Tr}(\pi_\rho \sigma)$$

Bob never infers the state to be σ *when it is* ρ

BUT

$$S_{\min}(\rho \| \sigma) := -\log \, \mathrm{Tr}\, \pi_\rho \sigma$$

Hence $\beta = 2^{-S_{\min}(\rho\|\sigma)}$

= *Prob(Type II error | Type I error = 0)*

- Compare with the operational significance of $S(\rho \| \sigma)$

  arises in asymptotic hypothesis testing

- Suppose Bob is given many (n) identical copies of the state

  - He receives $\rho^{\otimes n}$ or $\sigma^{\otimes n}$

- For n large enough,

  - $Prob(Type\ II\ error\ |\ Type\ I\ error < \varepsilon)$

$$\approx 2^{-n\, S(\rho\|\sigma)}$$

for any fixed $0 \leq \varepsilon \leq 1$

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- Hence, $$S_{\min}(\rho\|\sigma) \;\&\; S(\rho\|\sigma)$$

have similar interpretations in terms of *Prob(Type II error)*

$$S_{\min}(\rho\|\sigma):$$

- a single copy of the state
- *Prob(Type I error)* $= 0$

$$S(\rho\|\sigma):$$

- n copies of the state
- *Prob(Type I error)* $\underset{n\to\infty}{\longrightarrow} 0$

- Like $S(\rho \| \sigma)$ we have for ρ, σ states

$$S_*(\rho \| \sigma) \geq 0$$

for $* = \max, \min$

$$S_*(\Lambda(\rho) \| \Lambda(\sigma)) \leq S_*(\rho \| \sigma)$$

for any CPTP map Λ

- Also $$S_*(\rho \| \sigma) = S_*(U \rho U^* \| U \sigma U^*)$$ for any unitary operator U

- Most interestingly

$$S_{\min}(\rho \| \sigma) \leq S(\rho \| \sigma) \leq S_{\max}(\rho \| \sigma)$$

- The min-relative entropy is jointly convex in its arguments.

- The max-relative entropy is quasiconvex:

  For two mixtures of states $\rho = \sum_{i=1}^{n} p_i \rho_i$ & $\sigma = \sum_{i=1}^{n} p_i \sigma_i$

$$S_{\max}(\rho \| \sigma) \leq \max_{1 \leq i \leq n} S_{\max}(\rho_i \| \sigma_i)$$

- Also act as parent quantities for other entropies..........

Min- and Max- entropies

$$H_{\min}(\rho) := -S_{\max}(\rho \,\|\, I) = -\log \|\rho\|_{\infty}$$

$$H_{\max}(\rho) := -S_{\min}(\rho \,\|\, I) = \log \operatorname{rank}(\rho)$$

[Renner]

Just as:

von Neumann entropy

$$S(\rho) = -S(\rho \,\|\, I)$$

$$H_{\max}(\rho) \geq H_{\min}(\rho)$$

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- For a bipartite state ρ_{AB} :

$$H_{\min}(A|B)_\rho := -S_{\max}(\rho_{AB} \| I_A \otimes \rho_B)$$ etc.

just as: $$S(A|B) = -S(\rho_{AB} \| I_A \otimes \rho_B)$$

$$I_{\min}(A:B)_\rho := S_{\min}(\rho_{AB} \| \rho_A \otimes \rho_B)$$ etc.

just as: $$I(A:B) = S(\rho_{AB} \| \rho_A \otimes \rho_B)$$

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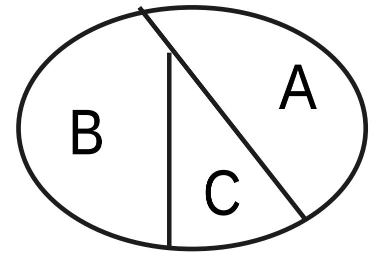
Min- and Max- Relative Entropies satisfy the:

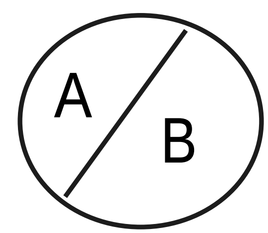


(1) *Strong Subadditivity Property*

$$H_{\min}(A|BC)_\rho \leq H_{\min}(A|B)_\rho$$

(2) *Subadditivity Property*

$$H_{\max}(\rho_{\mathrm{AB}}) \leq H_{\max}(\rho_{\mathrm{A}}) + H_{\max}(\rho_{\mathrm{B}})$$



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(Q) What are the operational significances of the min- and max- relative entropies in Quantum Information Theory?

A class of important problems

the evaluation of: optimal rates of info-processing tasks

- data compression,
- transmission of information through a channel
- entanglement manipulation **etc.**

Initially evaluated in the *"asymptotic, memoryless setting"*

under the following assumptions:

- information sources & channels were memoryless
- they were used an infinite number of times (asymptotic limit)

- Optimal rates -- entropic quantities

  obtainable from the relative entropy → *parent quantity*

- optimal rate of data compression:

  : the minimum number of qubits needed to store (compress) info emitted per use of a quantum info source : *reliably*

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Quantum Data Compression

Quantum Info source  *signals*

signals (*pure states*) $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_k\rangle \in \mathcal{H}$

with probabilities $p_1, p_2, \ldots, p_k$

Hilbert space

- Then source characterized by: $\{\rho, \mathcal{H}\}$

density matrix

$\{p_i, |\psi_i\rangle\}$

$$\rho = \sum_{i=1}^{k} p_i |\psi_i\rangle\langle\psi_i|$$

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To evaluate data compression limit :

Consider a sequence $\{\rho_n, \mathcal{H}_n\}_n$

If the quantum info source is memoryless

$$\mathcal{H}_n = \mathcal{H}^{\otimes n}; \qquad \boxed{\rho_n = \rho^{\otimes n}} \qquad \rho \in \mathcal{B}(\mathcal{H})$$

e.g. A memoryless quantum info source emitting qubits

- Consider n successive uses of the source ; n qubits emitted
- Stored in m_n qubits ; $m_n < n$ *(data compression)*

$$\boxed{\textit{rate of data compression} = \frac{m_n}{n}}$$

Asymptotic, Memoryless Setting

Optimal rate of data compression $R_\infty := \lim_{n\to\infty} \frac{m_n}{n}$

under the requirement that $p^{(n)}_{error} \underset{n\to\infty}{\to} 0$

$$R_\infty = S(\rho) \quad = - S(\rho \,\|\, I) \qquad \textit{(parent)}$$

- von Neumann entropy of the source $S(\rho) := -\mathrm{Tr}\,(\rho \log \rho)$

- WHAT IF : ρ_n: *state of a quantum spin system* (n *interacting spins)*

$\{\rho_n, \mathcal{H}_n\}_n$ $\rho_n \neq \rho^{\otimes n}$ *not memoryless !*

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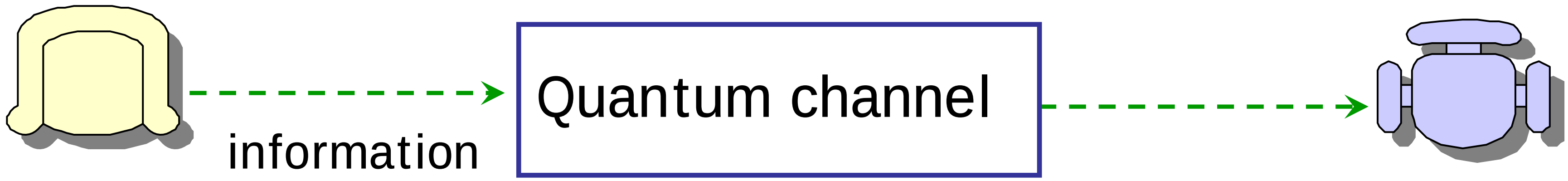
Transmission of information through a quantum channel



Examples:

- Optical fibre : through which polarized photons are transmitted

mobile particles which carry the info

- A quantum spin chain – governed by a suitable Hamiltonian
  - *info carriers (spin-1/2 particles) not mobile*
  - *instead the dynamical properties of the spin chain are exploited to transmit info*

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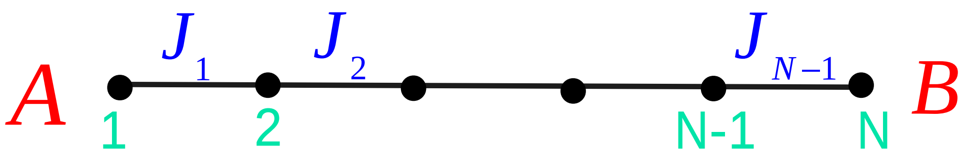
Perfect transfer of state through a quantum spin chain



[Christandl, ND, Ekert, Landahl]

$$H = \frac{1}{2}\sum_{i=1}^{N} J_i \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y\right)$$

with

$$J_i = \frac{\lambda}{2}\sqrt{i(N-i)}$$

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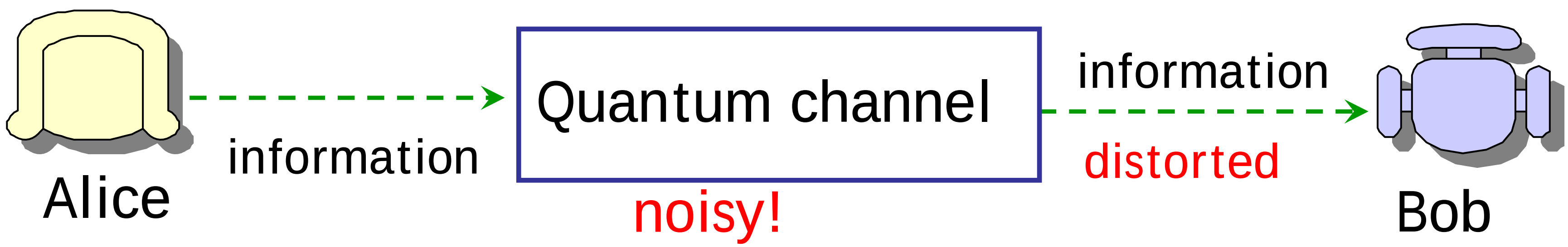
Transmission of information through a quantum channel



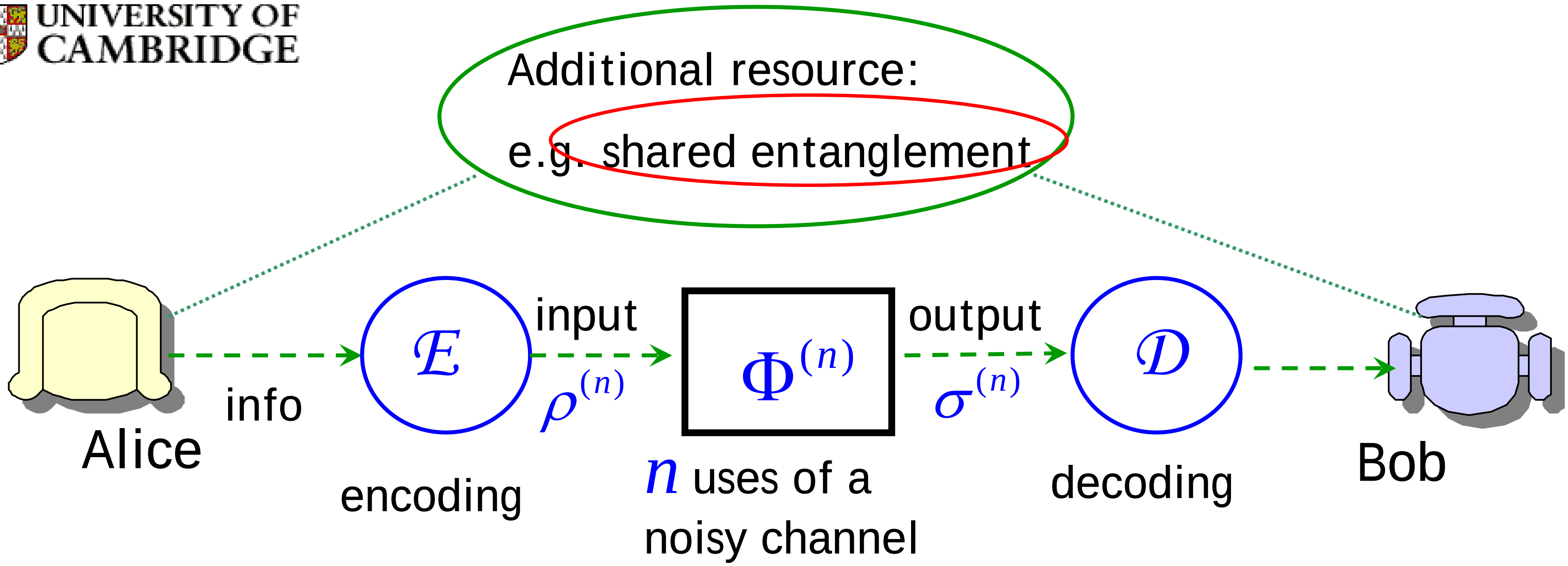
Optimal rate/capacity : the max. amount of info that can be reliably transmitted per use of the channel

- Let $\Phi^{(n)}$: n successive uses of a quantum channel Φ

Φ memoryless if: no correlation in the noise affecting states transmitted through successive uses of the channel:

$$\Phi^{(n)} = \Phi^{\otimes n}$$

$$p_e^{(n)} \xrightarrow[n\to\infty]{} 0$$



- Information: --- classical or quantum
- Input states:

-- product states $\rho^{(n)} = \rho_1 \otimes \rho_2 \otimes \dots \otimes \rho_n$

or

-- entangled states

- Measurements:
  - individual
  - collective

- These capacities evaluated in : *asymptotic, memoryless setting*

- *Parent quantity* = *quantum relative entropy*

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In real-world applications "asymptotic memoryless setting"

not necessarily valid

- **In practice:** info. sources & channels are used a finite number of times;
- there are unavoidable correlations between successive uses *(memory effects)*

e.g. "Spin chain model for correlated quantum channels"

Rossini et al, New J.Phys. 2008

Hence it is important to evaluate optimal rates for

finite number of uses (or even a single use)

of an arbitrary source, channel or entanglement resource

- Corresponding optimal rates:

⇒ *optimal one-shot rates*

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(Q) How can memory effects (effects of correlated noise) arise in a single use (of a source or channel) ?

(A) e.g. for a channel: We could have :

$$\Phi = \tilde{\Phi}^{(m)}$$

m uses of a channel $\tilde{\Phi}$ with memory
(finite)

- Hence, **one-shot capacity** encompasses the capacity of a channel for a finite number of its uses!

- scenario of practical interest!

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Min- & Max relative entropies: $S_{\min}(\rho\|\sigma),\ S_{\max}(\rho\|\sigma)$

act as parent quantities for one-shot rates of protocols

just as

Quantum relative entropy: $S(\rho\|\sigma)$

acts as a parent quantity for asymptotic rates of protocols

e.g. Quantum Data Compression $\{\rho, \mathcal{H}\}$ *memoryless source*

asymptotic rate: $S(\rho) \quad = -S(\rho\|I)$

more precisely

one-shot rate: $H_{\max}(\rho) \quad = -S_{\min}(\rho\|I)$

$H^{\varepsilon}_{\max}(\rho)$

[Koenig & Renner]

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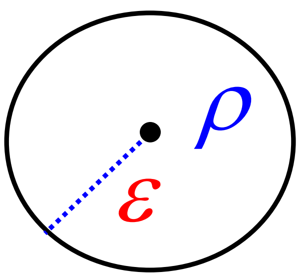
$$H^{\varepsilon}_{\max}(\rho)\ ?$$

$$\forall\ 0 < \varepsilon < 1$$

$H^{\varepsilon}_{\max}(\rho)$ { Optimal rate of *one-shot data compression* for a *maximum probability of error* $\leq \varepsilon$

smoothed max- entropy

$$H^{\varepsilon}_{\max}(\rho) := \min_{\bar{\rho} \in B^{\varepsilon}(\rho)} H_{\max}(\bar{\rho})$$



$$B^{\varepsilon}(\rho) := \{ \bar{\rho} :\ \| \rho - \bar{\rho} \|_1 \leq \varepsilon \}$$

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Further Examples

$$S_{\min}(\rho \| \sigma)$$

Parent quantity for the following:

- *[Wang & Renner] : one-shot classical capacity of a quantum channel*

- *[ND & Buscemi] : one-shot entanglement cost under LOCC*

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$$S_{\max}(\rho \| \sigma)$$

Parent quantity for the following:

- *[Buscemi & ND] : one-shot quantum capacity of a quantum channel*

- *[ND & Hsieh] : one-shot entanglement-assisted classical capacity of a quantum channel (today!)*

- *[Buscemi & ND] : one-shot entanglement distillation*

etc.

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Why are one-shot results important?

- One-shot results yield the known results of the

  asymptotic, memoryless case, on taking:

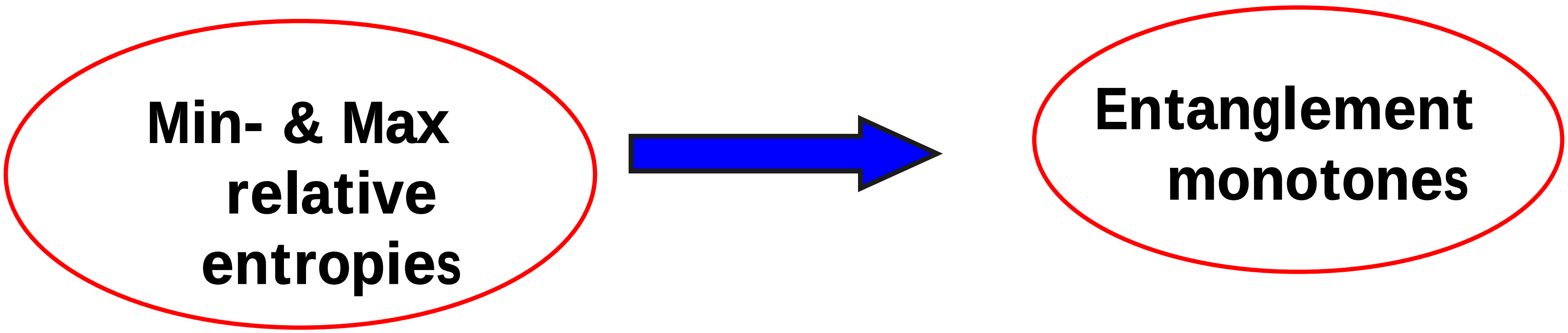
  $$n \rightarrow \infty \quad \text{and then} \quad \varepsilon \rightarrow 0$$

- Hence the one-shot analysis is more general !

- One-shot results also take into account effects of correlation (or memory) in sources, channels etc.

- In fact, one-shot results can be looked upon as the

  fundamental building blocks of Quantum Info. Theory


Min- & Max relative entropies
Entanglement monotones

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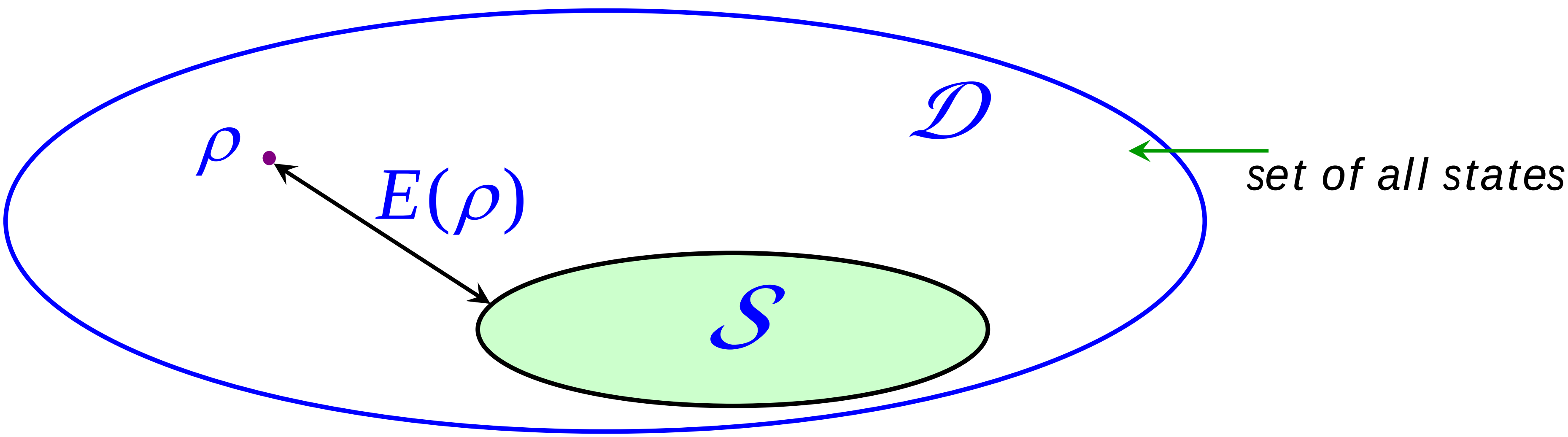
Entanglement monotones

- *Let* $\rho = \rho_{AB}$

> - $E(\rho)$ = *a measure of how entangled a state* ρ *is ;*
>
>   *i.e., the amount of entanglement in the state* ρ

: "minimum distance" of ρ *from the set* $\mathcal{S}$ *of separable states.*



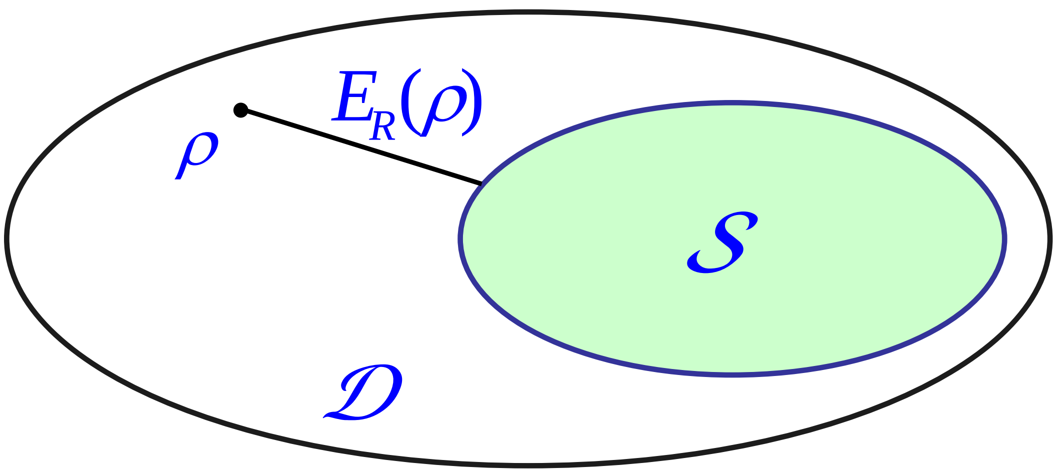
Relative Entropy of Entanglement

- One of the most important and fundamental entanglement measures for a bipartite state $\rho = \rho_{AB}$

$$E_R(\rho) := \min_{\sigma \in \mathcal{S}} S(\rho \| \sigma)$$

Quantum Relative Entropy

"distance"



Entanglement Monotones

$$E_R(\rho) = \min_{\sigma \in \mathcal{S}} S(\rho \| \sigma)$$

relative entropy of entanglement

- We can define two quantities:

$$E_{\max}(\rho) := \min_{\sigma \in \mathcal{S}} S_{\max}(\rho \| \sigma)$$

Max-relative entropy of entanglement

$$E_{\min}(\rho) := \min_{\sigma \in \mathcal{S}} S_{\min}(\rho \| \sigma)$$

Min-relative entropy of entanglement

these can be proved to be entanglement monotones!

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Properties of $E_{\max}(\rho)$, $E_{\min}(\rho)$

$* = \max,\ \min$

- $E_{*}(\rho) = 0$ *if* ρ *is separable*
- $E_{*}(\rho)$ is not changed by a local change of basis
- $E_{*}(\Lambda_{\mathrm{LOCC}}(\rho)) \leq E_{*}(\rho)$ *monotonicity*

(local operations & classical communication)

etc.

$$E_{\min}(\rho) \leq E_R(\rho) \leq E_{\max}(\rho)$$

$$\because \quad S_{\min}(\rho \| \sigma) \leq S(\rho \| \sigma) \leq S_{\max}(\rho \| \sigma)$$

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What are the *operational significances* *of*

$$E_{\min}(\rho) \;\&\; E_{\max}(\rho)\,?$$

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$E_{\max}(\rho)$ and $E_{\min}(\rho)$

have interesting operational significances in

entanglement manipulation

- *What is entanglement manipulation ?*

= Transformation of entanglement from one form to another by local operations & classical communication

(**LOCC) :**

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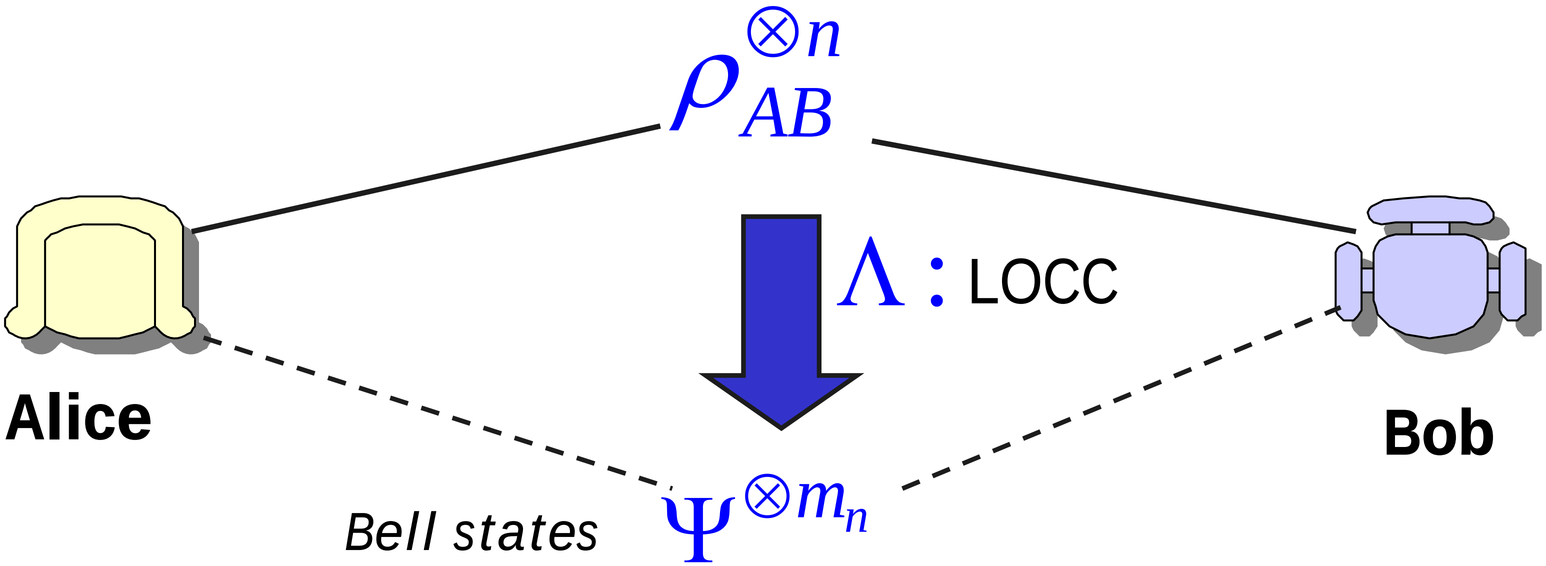
Entanglement Distillation



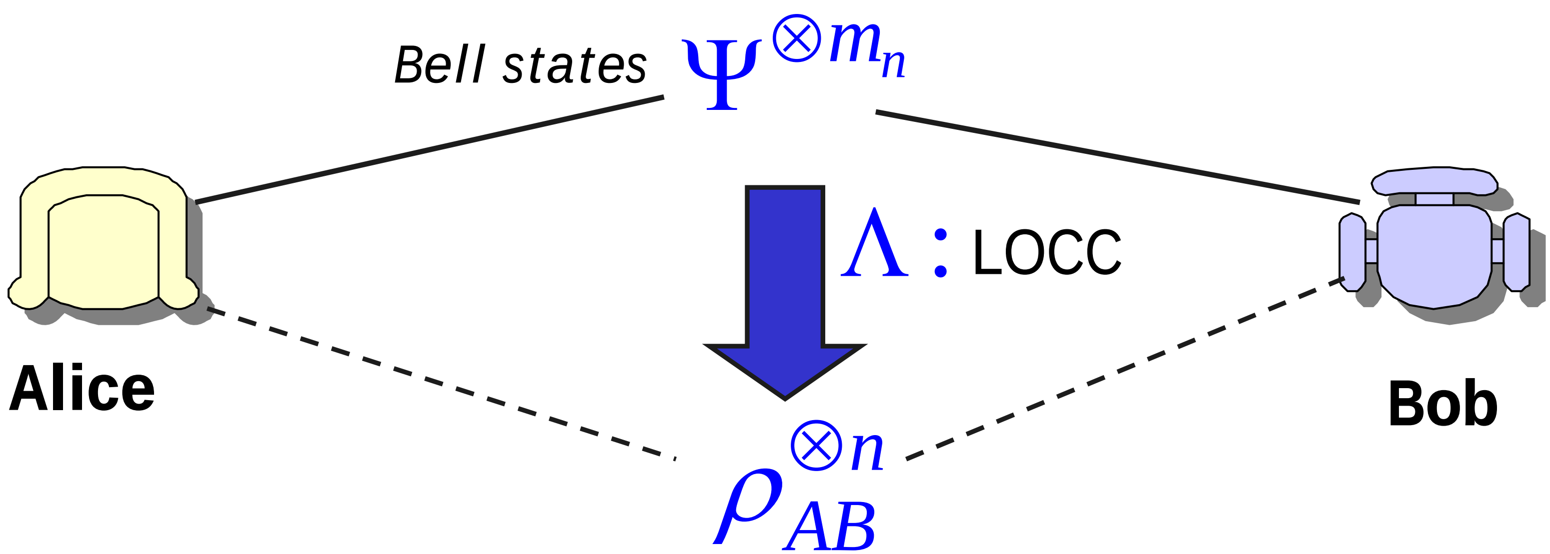
$$F\left(\Lambda\left(\rho_{AB}^{\otimes n}\right), \Psi^{\otimes m_n}\right) \xrightarrow[n\to\infty]{} 1$$

$\limsup_{n\to\infty} \frac{m_n}{n} =$ *the **maximum number** of **Bell states** that can be **extracted from** the state* ρ_{AB}

= *"distillable entanglement"*

Entanglement Dilution

- Bell states : resource for creating a desired target state



$$\liminf_{n\to\infty} \frac{m_n}{n} =$$

the minimum number of Bell states needed to create the state ρ_{AB}

= "*entanglement cost*"

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$$E_{\max}(\rho) \quad \text{and} \quad E_{\min}(\rho)$$

have interesting operational significances in

entanglement manipulation

when

~~$n \to \infty$~~ ⟶ *"one-shot" scenario* $(n=1)$

~~*LOCC*~~ ⟶ *SEPP maps*

(separability-preserving maps)

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Separability Preserving (SEPP) Maps

- The largest class of CPTP maps which when acting on a separable state yields a separable state

- If ρ_{AB} separable then $\Lambda_{\mathrm{SEPP}}(\rho_{AB}) \equiv$ separable

- *A SEPP map cannot create or increase entanglement*
  - *like a LOCC map !*

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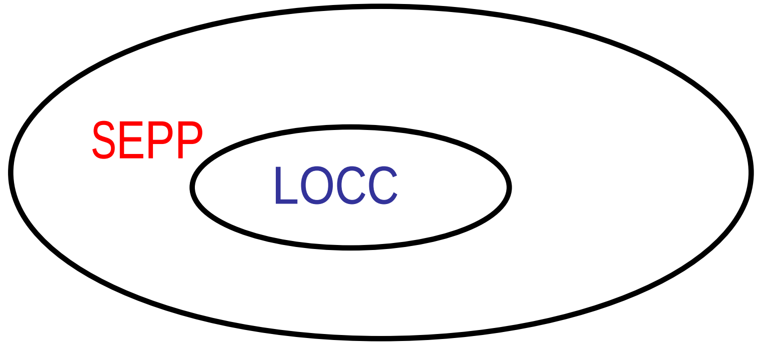
Separability Preserving (SEPP)

- Every LOCC operation is separability preserving
- BUT the converse is not true



- E.g. Consider the map :

$$\Lambda \equiv \Lambda_{AB}^{swap} = \text{ SWAP operation}$$

$$\Lambda_{AB}^{swap}\left(\sum_i p_i \, \rho_i^A \otimes \rho_i^B\right) = \sum_i p_i \, \rho_i^B \otimes \rho_i^A$$

separable state

- SWAP is not a local operation

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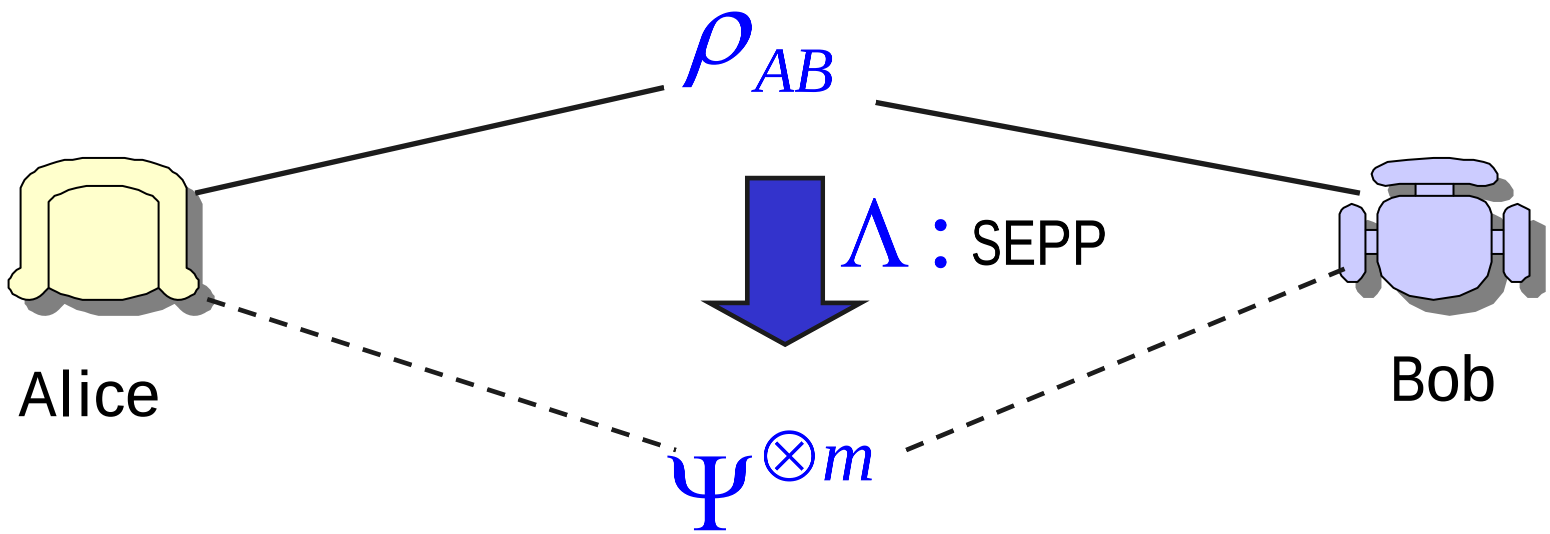
One-Shot Entanglement Distillation



- *What is the maximum number of Bell states that can be extracted from a single copy of* ρ_{AB} *using SEPP maps?*

i.e., what is the maximum value of m *?*

"*one-shot distillable entanglement* of ρ_{AB} "

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- *Result :*

"*one-shot distillable entanglement* of ρ_{AB}"

$$\approx E_{\min}(\rho_{AB})$$

ND &F.Brandao

Min-relative entropy of entanglement

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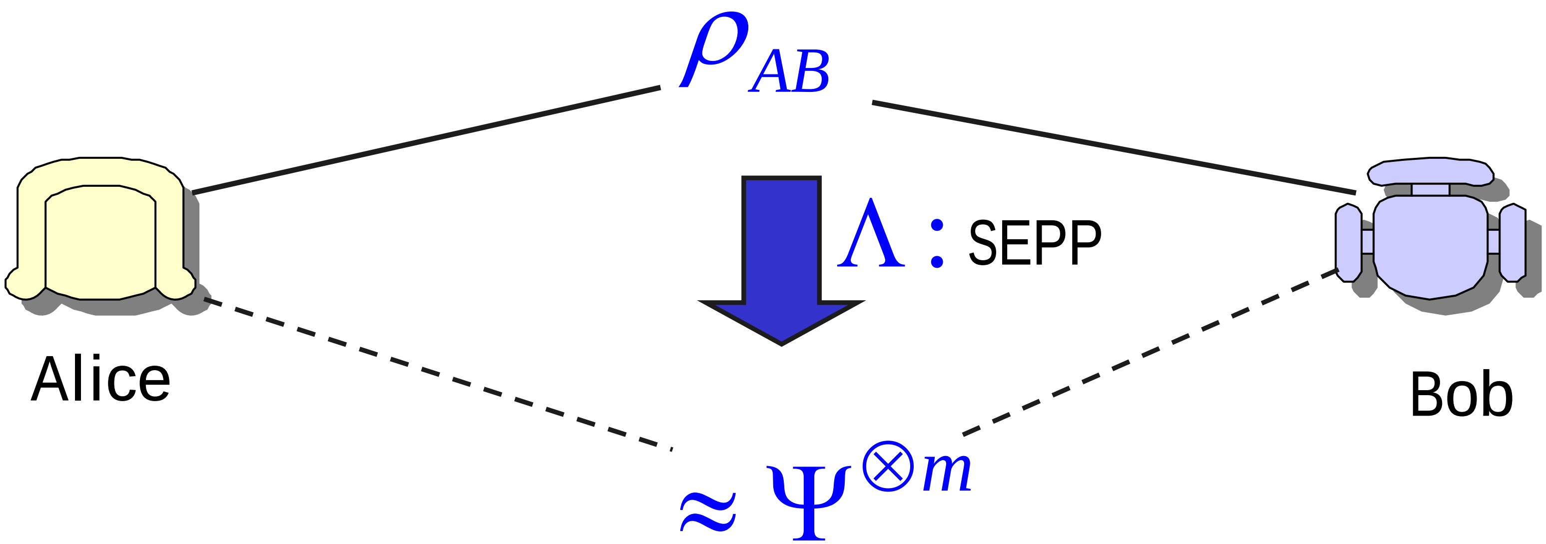
One-Shot Entanglement Distillation



$$F(\Lambda(\rho_{AB}), \Psi^{\otimes m}) \geq 1 - \varepsilon$$ *for some given* $\varepsilon > 0$

Then the maximum value of m :

One-shot ε *– error distillable entanglement*	$= E^{\varepsilon}_{\min}(\rho_{AB})$

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- where

$$E^{\varepsilon}_{\min}(\rho_{AB}) := \max_{\bar{\rho} \in B^{\varepsilon}(\rho_{AB})} E_{\min}(\rho_{AB})$$

$$B^{\varepsilon}(\rho) := \{\bar{\rho} : \ \|\rho - \bar{\rho}\|_1 \leq \varepsilon\}$$

smoothed min-relative entropy of entanglement

- similarly

$$E^{\varepsilon}{}_{\max}(\rho) := \min_{\bar{\rho} \in B_{\varepsilon}(\rho)} E_{\max}(\bar{\rho})$$

smoothed max-relative entropy of entanglement

(operational significance in entanglement dilution under SEPP maps)

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Summary

- Introduced 2 new relative entropies

  (1) *Min-relative entropy* & (2) *Max-relative entropy*

$$S_{\min}(\rho\|\sigma) \leq S(\rho\|\sigma) \leq S_{\max}(\rho\|\sigma)$$

- *Parent quantities for optimal one-shot rates for*
  - (i) data compression for a quantum info source
  - (ii) transmission of (a) classical info & (b) quantum info

    through a quantum channel
  - (iii) entanglement manipulation

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Entanglement monotones

- *Min-relative entropy of entanglement* $E_{\min}(\rho_{AB})$
- *Max-relative entropy of entanglement* $E_{\max}(\rho_{AB})$

- *Operational interpretations:*

$E^{\varepsilon}_{\min}(\rho_{AB})$: *One-shot distillable entanglement of* ρ_{AB} *under SEPP*

$E^{\varepsilon}_{\max}(\rho_{AB})$: *One-shot entanglement cost of* ρ_{AB} *under SEPP*

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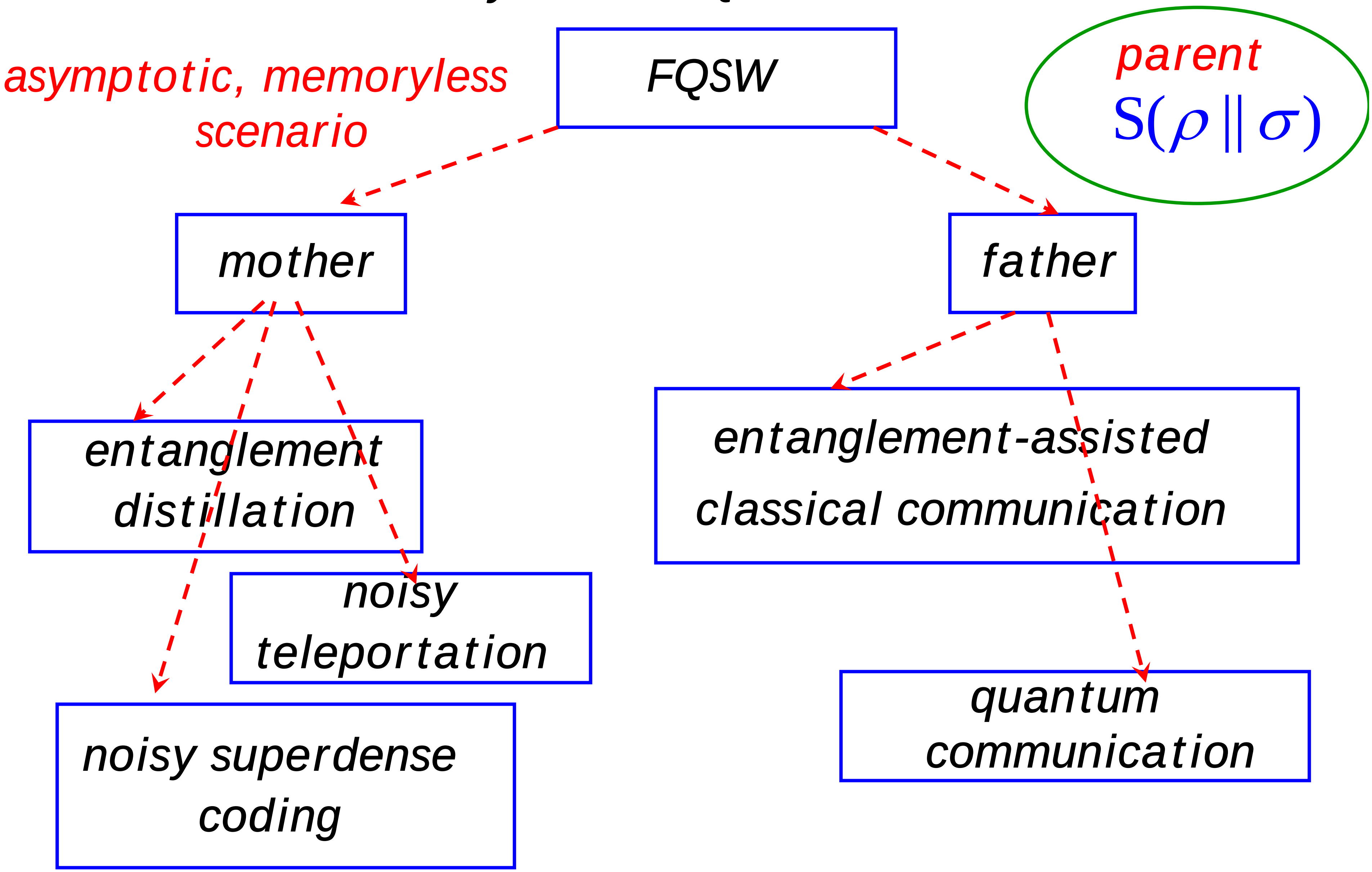
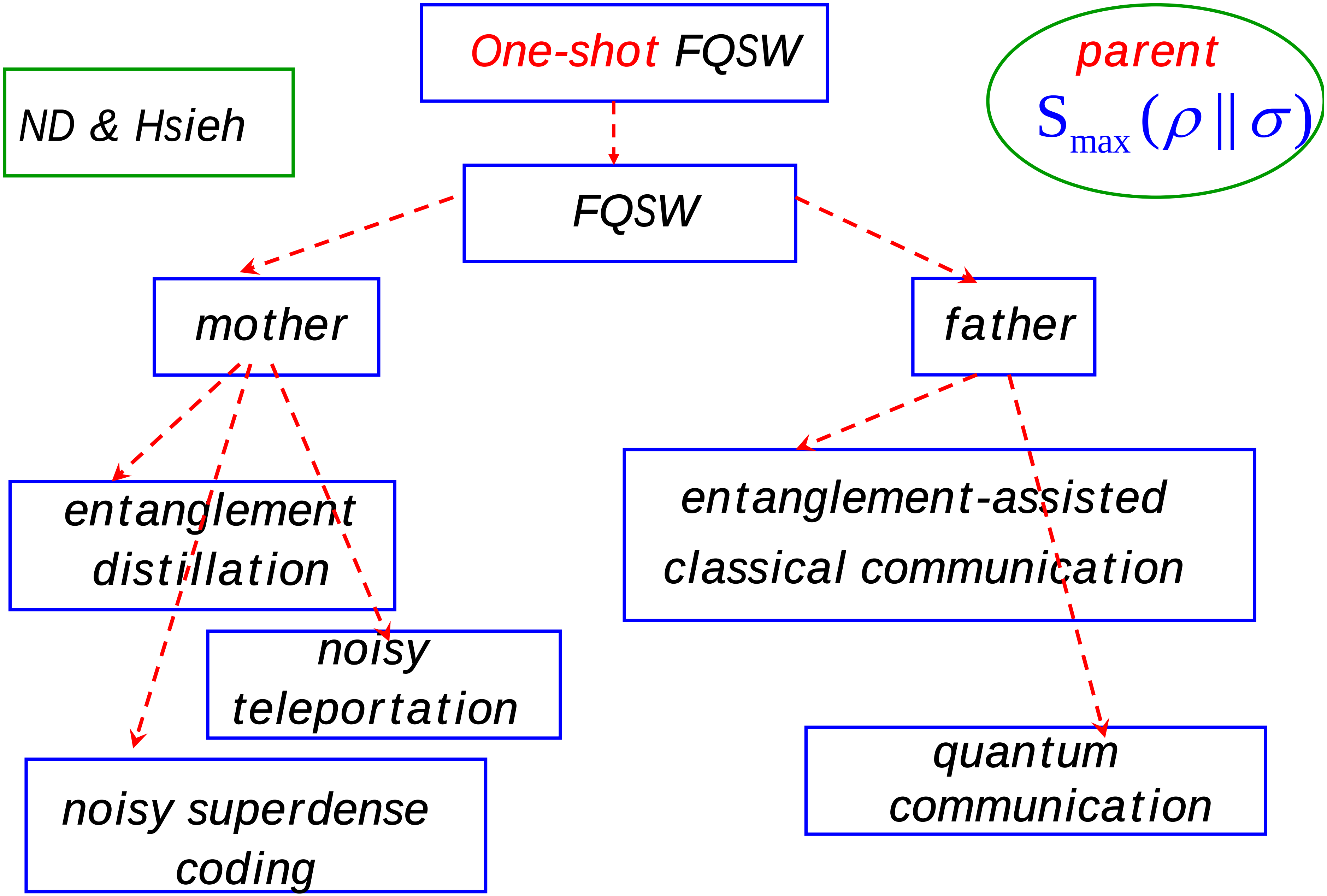
Family Tree of Quantum Protocols



Apex of the Family Tree of Quantum Protocols



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  - *Min-Hsiu Hsieh (Cambridge, UK)*

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