Towards stability of gapped quantum systems.

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May 19th, 2011
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The challenge...

``Find a minimal set of assumptions under which gapped Hamiltonians are stable against local perturbations.''

-Schrödinger's cat.

Isn’t every gapped system stable?

Counterexample to stability: Opening the gap.

Splitting the groundstate subspace.

Example

Consider 2-D ($N \times N$) Ising Hamiltonian and its perturbation:

$$H_N = \sum_{|i-j|=1} \frac{\mathbb{1} - \sigma_i^z \otimes \sigma_j^z}{2}, \quad H_N' = H_N - \frac{1}{N^2} \sum_{i=1}^{N^2} \sigma_i^z.$$

H_N has degenerate g.s. subspace spanned by $|000 \cdots 0\rangle$ and $|111 \cdots 1\rangle$, with spectral gap $\gamma_N = 1$, for all $N \geq 2$. H_N' has unique g.s. $|000 \cdots 0\rangle$, with $|111 \cdots 1\rangle$ now excited.
Bad quantum memory! The state $|+\rangle = |000 \ldots 0\rangle + |111 \ldots 1\rangle$ flips to $|-\rangle = |000 \ldots 0\rangle - |111 \ldots 1\rangle$ in time $t \sim \pi/2$, since $e^{itH_N'} |+\rangle = e^{-it} |000 \ldots 0\rangle + e^{it} |111 \ldots 1\rangle$.

Counterexample to stability: Closing the gap.

Low energy locally, but high energy globally.

Example

Consider a 2-D ($N \times N$) Ising Hamiltonian with a defect at the origin:

$$H_N = \frac{\mathbb{1} - \sigma_0^z}{2} + \sum_{|i-j|=1} \frac{\mathbb{1} - \sigma_i^z \otimes \sigma_j^z}{2}.$$

H_N has **unique, frustration-free** groundstate $|000\cdots 0\rangle$, with spectral gap $\gamma_N = 1$, for all $N \geq 2$. State $|111\cdots 1\rangle$ has same energy as groundstate everywhere, but at the origin. **Close the gap** by applying local operators everywhere, lowering the energy of $|111\ldots 1\rangle$, relative to $|000\ldots 0\rangle$. Use **local order parameter**, such as σ_i^z, as the perturbing term at each site.

$H'_N = H_N + \frac{1}{2N^2}\sum_{i=1}^{N^2} \sigma_i^z$ has degenerate g.s. $|000\cdots 0\rangle$ and $|111\cdots 1\rangle$.

Distinguishability implies instability!

Hamiltonians are **unstable** because local order parameters can act as perturbations to open the gap **between ground-states**, or close the gap **between ground-states and excited states** with low-energy, locally.

Projections onto local, low-energy eigenstates.

Definition

For Λ **the periodic lattice** $[-L, L]^d$, let $\mathbf{H_0} = \sum_{\mathbf{u}\in\mathbf{\Lambda}} \mathbf{Q_u}$, with each Q_u supported on $b_1(u)$, $u \in \Lambda$. Denote the **groundstate projector** by P_0 and define for $B = b_r(u)$, $r \leq L$, $u \in \Lambda$, **the projection** P_B **onto eigenstates** of $H_B = \sum_{b_1(u)\in B} Q_u$ with energy at most $\mathrm{Tr}(H_B P_0)$.

Stability needs...

Local Topological Quantum Order.

Local-TQO: For $\mathbf{A} = \mathbf{b_r}(\mathbf{u})$, $r \leq L^* \sim L^\alpha$, $\alpha \in (0,1]$, let O_A be an operator with support on A and define $\mathbf{A}(\ell) := \mathbf{b_{r+\ell}}(\mathbf{u})$. Then, $\mathbf{H_0}$ has **Local-TQO**, if there exists a **rapidly-decaying function** $\Delta_0(\ell)$, such that:

$$\|P_{A(\ell)} O_A P_{A(\ell)} - c_\ell(O_A) P_{A(\ell)}\| \leq \|O_A\| \, \Delta_0(\ell), \tag{1}$$

for $c_\ell(O_A) = \mathrm{Tr}(O_A \, P_{A(\ell)}) / \mathrm{Tr} \, P_{A(\ell)}$.

Note: The above condition implies that reduced density matrices to region A of states in $P_{A(\ell)}$ are identical up to error $\Delta_0(\ell)$.

Frustration-free Hamiltonians.

Definition

We say $\mathbf{H_0} = \sum_{\mathbf{u} \in \mathbf{\Lambda}} \mathbf{Q_u}$ has a **frustration-free** ground-state subspace P_0, if $Q_u P_0 = \lambda_u P_0$, where λ_u is the **smallest eigenvalue** of Q_u.

History of progress...

1. (Euclid, 314 B.C.) Let H_0 have spectral gap $\gamma > 0$ and **unique** groundstate. Then, $H_0 + V$ retains a gap if $\|V\| < \gamma/2$.

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3. (Bravyi, Hastings, M., '10) H_0 is sum of **commuting projections**, with spectral gap γ and **frustration-free** groundstate subspace, satisfying a form of **Local Topological Order**. Then, for V a sum of **rapidly decaying terms** V_u, there exists a J_0 such that for $\|V_u\| \le J_0 \implies$ **stable gap**.

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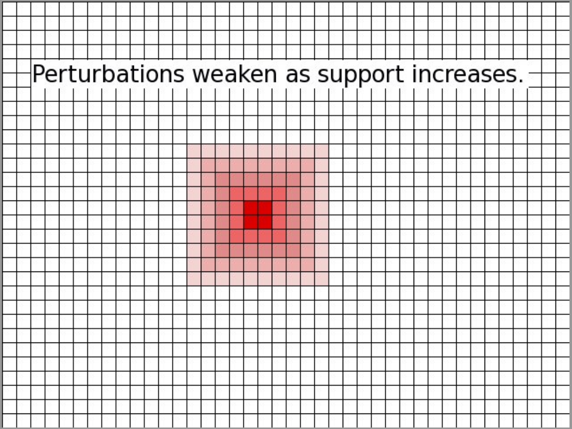
4. (M., Pytel, '11) Let H_0 have gap γ and **frustration-free** groundstate subspace, satisfying **Local Topological Order**. Then, stability holds for all perturbations V, as above. This talk.

Decaying perturbations...



For each site $u \in \Lambda$, we allow perturbations supported on $b_r(u)$. As the radius of the support increases, the norm of the perturbation decreases rapidly.

The Perturbations: Local decomposition and strength.

Definition

We say that V **has strength** J **and rapid decay** f, if we can write

$$V = \sum_{u \in \Lambda} V_u, \quad V_u := \sum_{r \geq 0} V_u(r),$$

such that $V_r(u)$ has support on $b_r(u)$ and $\|V_r(u)\| \leq J\, f(r)$, $r \geq 0$.

Frustration-free Hamiltonians are stable!

The spectral gap behaves as it should...

For a very general class of perturbations, frustration-free Hamiltonians with local topological order maintain a spectral gap even when the strength of each local perturbation increases to a constant independent of the system size!

Local Gaps.

Definition

Local-Gap: We say that $\mathbf{H_0}$ is **locally gapped** w.r.t. a function $\gamma(r)$, if $\mathbf{H_B} \geq \gamma(\mathbf{r})(\mathbf{1} - \mathbf{P_B})$, where $B = b_r(u)$.

Open Problem: Is this condition always satisfied with $\gamma(r)$ decaying at most polynomially, if H_0 is a sum of projections with frustration-free groundstate?

Open Problem 2: Is this condition really necessary?

The main result.

Theorem

- *Let* H_0 *be a* **frustration-free** *Hamiltonian satisfying* **Local-TQO** *and* **Local-Gap** *with decay given by* $\Delta_0(r)$ *and* $\gamma(r)$*, respectively.*

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- *Assume* **periodic-boundary conditions** *and a* **spectral gap** $\gamma > 0$.
- *Let V be a strength J perturbation, with decay given by $f(r)$.*
- *Then, $H_0 + V$ has spectral gap bounded below by*

$$(1 - c_0 J)\gamma - c_1\, J\, L^d \sqrt{\Delta_0(L^*)},$$

where

$$c_0 = \sum_{r=1}^{L} r^d \cdot \frac{w(r)}{\gamma(r)}$$

and $w(r) = \sum_{s=r}^{L^*} s\, f_1(s/4) + \|f_1\|_1 \sum_{s=r}^{L^*} \sqrt{\Delta_0(s/2)}$. The function f_1 is obtained from decay properties of f (Lieb-Robinson bounds).

Overview of the proof...

The 4 main steps.

- Using the **spectral flow**, unitarily transform the gapped family of Hamiltonians $H_0 + sV$ into $U^\dagger(s)(H_0 + sV)U(s) = H_0 + V'$, so that $[V', P_0] = 0$. Write $V' = W + \Delta + \mathrm{Tr}(P_0 V')\mathbb{1}$, where $\Delta = P_0 V' P_0$ and $W = (1 - P_0)V'(1 - P_0)$. (global block-diagonality)

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- **Relative boundedness** implies that $H_0 + W + \Delta$, has a spectral gap, which is equivalent to the stability of the spectrum of $H_0 + V$. (unitary invariance + global energy shift)

The Bravyi-Hastings bootstrapping argument.

Proof of Stability from Relative Bound.

- Assume that $s^* < 1$ is the largest s, such that $H_0 + sV$ maintains a gap at least $\gamma/2$, for $0 \leq s \leq s^*$ (γ is spectral gap of H_0).

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- If $P_0(s)\,|\Psi_0(s)\rangle = |\Psi_0(s)\rangle$ is an eigenvector of $H_0 + sV$ with eigenvalue $E_0(s)$, then $|\Psi_0(s)\rangle = U(s)\,|\Psi_0\rangle$, where $P_0\,|\Psi_0\rangle = |\Psi_0\rangle$.

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- We have: $U^\dagger(s)(H_0 + sV - E\cdot\mathbb{1})U(s)\,|\Psi_0\rangle = U^\dagger(s)(H_0 + sV - E\cdot\mathbb{1})\,|\Psi_0(s)\rangle = (E_0(s) - E)\,|\Psi_0\rangle$. Recalling that $H_0 + W + \Delta = U^\dagger(s)(H_0 + sV - E\cdot\mathbb{1})U(s)$, with $W\,P_0 = 0$, we also have:

$$(H_0 + W + \Delta)\,|\Psi_0\rangle = \Delta\,|\Psi_0\rangle = (E_0(s) - E)\,|\Psi_0\rangle\,.$$

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$$(H_0 + W + \Delta)\,|\Psi_0\rangle = \Delta\,|\Psi_0\rangle = (E_0(s) - E)\,|\Psi_0\rangle\,.$$

- Hence, $|E_0(s) - E| \leq \|\Delta\| << 1$ as the size of our lattice increases, which implies that all groundstates of $H_0 + W + \Delta$ have energy at most $\|\Delta\|$ and span P_0.

Proof of Stability from Relative Bound.

- Consider any state $|\psi_1\rangle$ orthogonal to P_0. Obviously, $U(s)|\psi_1\rangle$ will be orthogonal to $P_0(s) = U(s)P_0(0)U^\dagger(s)$, the ground state subspace of $H_0 + sV - E \cdot \mathbb{1}$.

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- What is the energy of $|\psi_1\rangle$ in $H_0 + W + \Delta$? Here is a lower bound:

$$\langle\psi_1| H_0 + W + \Delta |\psi_1\rangle \geq$$
$$\langle\psi_1| H_0 |\psi_1\rangle - |\langle\psi_1| W |\psi_1\rangle| - |\langle\psi_1| \Delta |\psi_1\rangle|$$
$$\geq (1 - c_0 J)\gamma - \|\Delta\|.$$

Hence, the gap of $H_0 + sV$ is at least $(1 - c_0 J)\gamma - 2\|\Delta\|$.

$\square$

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$$\langle\psi_1| H_0 + W + \Delta |\psi_1\rangle \geq$$
$$\langle\psi_1| H_0 |\psi_1\rangle - |\langle\psi_1| W |\psi_1\rangle| - |\langle\psi_1| \Delta |\psi_1\rangle|$$
$$\geq (1 - c_0\, J)\gamma - \|\Delta\|.$$

  Hence, the gap of $H_0 + sV$ is at least $(1 - c_0\, J)\gamma - 2\|\Delta\|$.
- Choosing J small enough, gap is made larger than $\gamma/2$! But, for $s^* + \epsilon$, the gap is smaller than $\gamma/2$. The contradiction must be that we assumed $s^* < 1$, for given strength $J \leq J_0 \sim 1/c_0$. So, $s^* = 1$.

□

The end.

Thank you!

Generators of quasi-adiabatic evolution (Hastings)

Definition

For $H_s = H_0 + sV$, define the quasi-adiabatic evolution generator $\mathcal{D}_s$ by:

$$\mathcal{D}_s \equiv \int_{-\infty}^{\infty} s_\gamma(t) \left(\int_0^t e^{iuH_s}(V) e^{-iuH_s} \mathrm{d}u \right) \mathrm{d}t, \tag{2}$$

where the function $s_\gamma(t)$ (called a **filter function**) is chosen to satisfy the following properties:

1. First, the Fourier transform of $s_\gamma(t)$, which we denote $\tilde{s}_\gamma(\omega)$, obeys

$$|\omega| \geq \gamma/2 \quad \rightarrow \quad \tilde{s}_\gamma(\omega) = 0 \quad \text{(compact support)}. \tag{3}$$

2. Second, $s_\gamma(t)$ decays like $\exp\{-\frac{\gamma|t|}{4\log^2 \gamma|t|}\}$ (sub-exponential decay).
3. Third, $s_\gamma(t) \geq 0$, so that $\mathcal{D}_s$ is Hermitian.
4. Note: This magical function $s_\gamma(t)$ exists and can be quite the ice-breaker on a first date.

Quasi-adiabatic evolution

Definition

Define a unitary operator U_s by

$$(\partial_{s'} U_{s'})_{s'=s} \equiv i\mathcal{D}_s U_s, \quad U_0 = \mathbb{1}. \tag{4}$$

Lemma

Let H_s be a differentiable family of Hamiltonians.
Let $P(s)$ denote the projection onto the eigenstates of H_s with energies in $[E_{min}(s), E_{max}(s)]$, where these energies are continuous functions of s.
Assume that all eigenvalues of H_s are either in the interval $[E_{min}(s), E_{max}(s)]$, or are separated by at least $\gamma/2$ from this interval.
Then, for all s with $0 \le s \le 1$, we have

$$P(s) = U_s P(0) U_s^\dagger. \tag{5}$$