Hermitian and Unitary Representations for Affine Hecke Algebras

Dan Barbasch

(joint with D. Ciubotaru)

May 2013

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Introduction

This talk is about aspects of representation theory of $p-$adic groups that parallel real groups. This conforms to the Lefschetz principle which states that what is true for real groups is also true for p-adic gorups.

In the case of real groups, the results refer to J. Adams, P. Trapa, M. vanLeuwen, W-L. Yee and D. Vogan on the one hand, Schmid and Vilonen on the other hand.

A major technique that does not apply in the p-adic case, is tensoring with finite dimensional representations. Different geometry plays a role in the representation theory of p-adic groups, results have been developed by Lusztig, Kazhdan-Lusztig and Ginzburg.

This talk will not have much geometry in it, mostly using the aforementioned results.

Many of the basic results quoted are standard for $p-$adic groups. But as the title indicates, the context is that of the affine graded Hecke algebra. The aim is to develop a self contained theory for graded affine algebras.

Most of the results presented follow [B], [BC1], [BC2], [BM3].

This is (still) work in progress.

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The Unitarity Problem, P-adic Groups

NOTATION

1. G is the rational points of a linear connected reductive group over a local field $\mathbb{F} \supset \mathcal{R} \supset \mathcal{P}$.

2. The Hecke algebra is

   $$\mathcal{H}(G) := \{f : G \longrightarrow \mathbb{C}, f \text{ compactly supported, locally constant } \}$$

3. A representation (π, U) is called hermitian if U admits a hermitian invariant form, and unitary, if U admits a $G-$invariant positive definite inner product.

4. It is called admissible if $Stab_G(v)$ for any vector $v \in U$ is open, and $U^{\mathcal{K}}$ is finite dimensional for any compact open subgroup $\mathcal{K} \subset G$.

Examples of such groups are isogeny forms of $SL(2)$, which we might call $Sp(2)$ and $SO(3)$, rational points of the simply connected and adjoint form. Another well studied example is $GL(2)$.

Compact open subgroups:

$$K_n = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \ : \ a,b,c,d \in \mathcal{R}, \ ad - bc = 1, a,d \in \mathcal{R}^\times, b,c \in \varpi^n \mathcal{R} \right\}$$

$K_0(=K)$ is a maximal compact open subgroup, in $GL(2)$ unique up to conjugacy. $SL(2)$ has another conjugacy class of maximal compact subgroups $K_0' = \begin{bmatrix} \varpi & 0 \\ 0 & 1 \end{bmatrix} \cdot K_0 \cdot \begin{bmatrix} \varpi^{-1} & 0 \\ 0 & 1 \end{bmatrix}$.

Iwahori subgroup $\mathcal{I} \subset K_0$:

$$\mathcal{I} := \left\{ \begin{bmatrix} a & b \\ \varpi c & d \end{bmatrix} \ : \ a,b,c,d \in \mathcal{R}, \ ad - \varpi bc = 1 \right\}$$

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The prime example of an admissible representation is the principal series $X(\chi)$ and its composition factors.

- $B = AN = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix} \right\} \cdot \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \right\}$

- Let $\chi \in \widehat{A}$. Define $(\pi_\chi, X(\chi))$
  $X(\chi) = \{ f : G \longrightarrow \mathbb{C} \ : \ f(gb) = \chi(a)^{-1} \delta(a)^{-1/2} f(g) \}$ with f
  locally constant and δ the modulus function, and
  $\pi_\chi(g) f(h) := f(g^{-1}h)$.

Because $G = KB$, this is an admissible representation.

Special Case: χ satisfying $\chi \mid_{A \cap K_0} = trivial$, is called unramified.
We write $\chi(a) = |\alpha|^\nu$, and $X(\nu)$.

Representations which are factors of $X(\nu)$ are called unramified. In particular the spherical representations, those which satisfy $V^{K_0} \neq (0)$ are unramified.

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PROBLEM

Classify all irreducible unitary representations of G.

It is enough (Harish-Chandra) to solve an

ALGEBRAIC PROBLEM:

Classify the unitary dual for irreducible admissible $\mathcal{H}(G)-$modules.

$\mathcal{H}(G)$ is an algebra under convolution, and is endowed with a conjugate linear involutive anti-automorphism $\star$,

$$(f)^{\star}(x) := \overline{f(x^{-1})}.$$

Hermitian: $\langle \pi(f)v_1, v_2 \rangle = \langle v_1, \pi(f^{\star})v_2 \rangle$.

Unitary: Hermitian plus $\langle \ , \ \rangle \gg 0$ *(i.e. positive definite)*.

A Reduction

According to results of Bernstein, the category of admissible representations breaks up into blocks. Further results, starting with Borel-Casselmann, Howe-Moy, Bushnell-Kutzko and many others (J. Kim, J.-K. Yu, ...), imply that each block is equivalent to a category of finite dimensional representations of an Iwahori-Hecke type algebra (or at least true for most blocks).

Main Example: (Borel-Casselmann)

(the prototype of the results mentioned above).

- G split, $B = AN \subset G$ a Borel subgroup.

- $\mathcal{I} \subset G$ an Iwahori subgroup

- $\mathcal{H}(\mathcal{I}\backslash G/\mathcal{I})$ the Iwahori-Hecke algebra of $\mathcal{I}-$biinvariant functions in $\mathcal{H}(G)$.

Theorem (Borel-Casselmann). *The category of admissible representations all of whose subquotients are generated by their $\mathcal{I}-$invariant vectors is equivalent to the category of finite dimensional $\mathcal{H}(\mathcal{I}\backslash G/\mathcal{I})-$modules via the functor $U \mapsto U^{\mathcal{I}}$.*

The functor takes a unitary module to a unitary module. But it is not at all clear why $U^{\mathcal{I}}$ unitary should imply U unitary.

Theorem (B-Moy). *A module (π, U) is unitary if and only if $(\pi^{\mathcal{I}}, U^{\mathcal{I}})$ is unitary.*

An ingredient of the proof is the independence of tempered characters, which is a consequence of results of Lusztig and Kazhdan-Lusztig, which depend on geometric methods.

The algebra $\mathcal{H}(\mathcal{I} \backslash G / \mathcal{I})$ can be described by generators and relations. In the case of a more general block, the analogous algebra to the one appearing in the Borel-Casselmann result is more complicated. Most (if not all) cases are covered by a generalization of the B-Moy theorem in [BC1].

The spherical unitary dual for split groups is completely known, [BM3], [B], [C], [BC], ...

The unitary dual for $p-$adic $GL(n)$ is known by work of Tadic (much earlier). Other groups of type A are also known, *e.g.* division algebras, work of Secherre.

These examples can be made to fit in the general program outlined earlier, *i.e.* use the blocks to reduce the problem to the analogous one for affine graded Hecke algebras, and solve that problem instead.

Hecke algebra for type A_1

$\mathcal{H}(\mathcal{I}\backslash G/\mathcal{I})$ is generated by θ, T satisfying

$$T^2 = (q-1)T + q$$

and

- $T\theta = \theta^{-1}T + (q-1)(\theta+1)$ $(Sp(2))$

- $T\theta = \theta^{-1}T + (q-1)\theta$ $(SO(3))$

Note: Because the category of representations breaks up according to infinitesimal character, several affine graded algebras are needed in order to compute the unitary dual. This involves a reduction to real infinitesimal character; it is analogous to the real case, but in fact more general. We will assume it at some point.

The Graded Affine Hecke Algebra

Notation:

- $\Phi = (V, R, V^\vee, R^\vee)$ an $\mathbb{R}-$root system, reduced.

- W the Weyl group.

- $\Pi \subset R$ simple roots, R^+ positive roots.

- $k : \Pi \to \mathbb{R}$ a function such that $k_\alpha = k_{\alpha'}$ whenever $\alpha, \alpha' \in \Pi$ are W-conjugate.

Definition (Graded Affine Hecke Algebra).
$\mathbb{H} = \mathbb{H}(\Phi, k) \cong \mathbb{C}[W] \otimes S(V_\mathbb{C})$ *such that*

(i) $\mathbb{C}[W]$ *and* $S(V_\mathbb{C})$ *have the usual algebra structure,*

(ii) $\omega t_{s_\alpha} = t_{s_\alpha} s_\alpha(\omega) + k_\alpha \langle \omega, \check{\alpha} \rangle$ *for all* $\alpha \in \Pi$, $\omega \in V_\mathbb{C}$.

Star Operations

In order to be able to consider hermitian and unitary modules for an algebra $\mathbb{H}$, we need a star operation; a conjugate linear involutive algebra anti-automorphism κ.

(π, U) gives rise to (π^κ, U^h) by the formula

$$\left(\pi^\kappa(h)f\right)(v) := f(\pi(\kappa(h))v)$$

(π, U) admits a $\kappa-$invariant sesquilinear form if and only if there is a (nontrivial $\mathbb{C}-$linear) equivariant map $\iota : (\pi, U) \longrightarrow (\pi^\kappa, U^h)$. Define

$$\langle h_1, h_2 \rangle := \iota(h_1)(h_2).$$

The form is **hermitian** if $\iota^h : U \subset \left(U^h\right)^h \longrightarrow U^h$ coincides with ι.

Note: This is already simpler than the real case because we are dealing with finite dimensional representations ($\subset$ is $=$).

$\mathbb{H}$ has a natural κ which we will denote by $\bullet$:

$$\text{\color{red}$(t_w)^\bullet = t_{w^{-1}}, \qquad (\omega)^\bullet := \overline{\omega}, \ \omega \in V_{\mathbb{C}}.$}$$

(Recall that $V_{\mathbb{C}}$ is the complexification of the real vector space V).
Bullet is an involutive anti-automorphism because

$$(t_\alpha \omega)^\bullet = \omega^\bullet t_\alpha^\bullet = \overline{\omega} t_\alpha = t_\alpha \overline{s_\alpha(\omega)} + \langle \overline{\omega}, \check{\alpha} \rangle$$

while

$$(s_\alpha(\omega) t_\alpha + \langle \omega, \check{\alpha} \rangle)^\bullet = t_\alpha^\bullet s_\alpha(\omega)^\bullet + \overline{\langle \omega, \check{\alpha} \rangle} = t_\alpha \overline{s_\alpha(\omega)} + \langle \overline{\omega}, \check{\alpha} \rangle.$$

However if $\mathbb{H}$ is obtained from a $p-$adic group, the star
$f^\star(x) := \overline{f(x^{-1})}$ induces a $\star$ on $\mathbb{H}$, which is NOT $\bullet$.

It is not far off though; the κ coming from the group has to satisfy

(i) $\kappa(t_w) = t_{w^{-1}}$,

(ii) $\kappa(V_\mathbb{C}) \subset \mathbb{C}[W] \cdot V_\mathbb{C}$.

Condition (ii) is analogous to the case of a real group. κ is required to preserve $\mathfrak{g} \subset U(\mathfrak{g})$, so it comes down to classifying real forms of $\mathfrak{g}$.

Theorem (B-Ciubotaru). *Assume the root system Φ is simple. The only involutive antiautomorphisms κ satisfying (i) and (ii) are*

$\bullet$ *from before,*

and

$\star$*, determined by* $\omega^\star = t_{w_0}\overline{(-w_0\omega)}t_{w_0}$*, where $w_0 \in W$ is the long Weyl group element.*

We define $a : \mathbb{H} \longrightarrow \mathbb{H}$ to be the automorphism determined by

$$a(t_w) := t_{w_0 w w_0}, \qquad a(\omega) := -w_0(\omega).$$

Then $a(h^\bullet) = (a(h))^\bullet$ and $\star = \operatorname{Ad} t_{w_0} \circ a \circ \bullet$.

In all the examples we know, $\star$ is the star operation coming from the group.

The underlying reason for this discrepancy is that

$$(\delta_{\mathcal{I}a\mathcal{I}})^{-1} \ \textcolor{red}{\neq} \ \delta_{\mathcal{I}a^{-1}\mathcal{I}} \quad \text{for all } a \in A.$$

An example is provided in $SL(2)$ by $a = \begin{bmatrix} \varpi & 0 \\ 0 & \varpi^{-1} \end{bmatrix}$.

It is true however that

$$(\delta_{\mathcal{I}w\mathcal{I}})^{-1} = (\delta_{\mathcal{I}w\mathcal{I}}) \quad \text{for } w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Sketch of Proof

κ an involutive automorphism, $\kappa(t_w) = t_w$ and (ii) as before.

1. $\kappa(t_w) = t_w, \quad w \in W$; $\kappa(\omega) = c_0\omega + \sum_{y \in W} g_y(\omega)t_y, \quad \omega \in V_\mathbb{C}$,
    where $g_y : V_\mathbb{C} \to \mathbb{C}$, $y \in W$, are linear.

2. $t_{s_\alpha}\omega - s_\alpha(\omega)t_{s_\alpha} = k_\alpha(\omega, \alpha^\vee)$ implies for all $\omega \in V_\mathbb{C}$, $\alpha \in \Pi$,

    $$g_{s_\alpha y s_\alpha}(\omega) = \begin{cases} g_y(s_\alpha(\omega)), & y \neq s_\alpha, \\ g_{s_\alpha}(s_\alpha(\omega)) + k_\alpha(1 - c_0)(\omega, \alpha^\vee), & y = s_\alpha. \end{cases}$$

3. $\kappa^2 = Id$ implies $c_0^2 = 1$. If $c_0 = 1$, $g_y = 0$, so $\kappa = Id \leftrightarrow \bullet$.

    If $c_0 = -1$ and $a = Id$, $\mathrm{Ad}\, t_{w_0} \circ \kappa$ is another κ, but has $c_0 = 1$.
    So $\kappa = \mathrm{Ad}\, t_{w_0} \leftrightarrow \star$.

If $a \neq Id$ need another page of computations.

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Strategy

In order to classify the unitary dual one needs to know first which irreducible modules are hermitian.

- Classify all admissible irreducible modules

- Single out the hermitian ones

For the affine graded algebra admissible means finite dimensional.

Langlands Classification

Definition. *A module (σ, U) is called tempered (modulo the center) if all the weights η of $V_{\mathbb{C}}$ satisfy $Re\langle \varpi_{\alpha}, \eta \rangle \leq 0$ for all $\alpha \in \Pi$, ϖ_{α} the corresponding fundamental weight.*

Let $\Pi_M \subset \Pi$ be a subset of the simple roots.

- $\mathbb{H}_M := span\{t_{\alpha}, \ \omega\}$, $\alpha \in \Pi_M$, $\omega \in V_{\mathbb{C}}$.

- $V_M \subset V$ the kernel of the $\check{\alpha}$ with $\alpha \in \Pi_M$.

- $X(M, \sigma_0, \nu) := \mathbb{H} \otimes_{\mathbb{H}_M} [U_{\sigma_0} \otimes \mathbb{C}_{\nu}]$
  the standard module attached to a tempered module σ_0 of (the semisimple part of) $\mathbb{H}_M$ and a character ν of V_M.

Theorem (Langlands Classification, *cf* [Ev]).

(i) *If $Re\langle \nu, \alpha \rangle > 0$ for all $\alpha \in \Pi \backslash \Pi_M$, then $X(M, \sigma_0, \nu)$ has a unique irreducible quotient $L(M, \sigma_0, \nu)$.*

(ii) *Every irreducible module is isomorphic to an $L(M, \sigma_0, \nu)$.*

(iii) *$L(M, \sigma_0, \nu) \cong L(M', \sigma_0', \nu')$ if and only if $M = M'$, $\sigma_0 \cong \sigma_0'$, $\nu = \nu'$.*

Denote by w^0 the minimal element in $w_0 \, aM$, and $w^0 \sigma_0$, $w^0 \nu$ the transfers of σ_0, ν to $\mathbb{H}_{w_0 M}$

Then $L(M, \sigma_0, \nu)$ is the image of an intertwining operator

$$\color{blue} A_{w^0} : X(M, \sigma_0, \nu) \longrightarrow X(aM, w^0 \sigma_0, w^0 \nu),$$

$$\color{blue} h \otimes v \mapsto h R_{w^0} \otimes w_0(v)$$

$R_{w^0} \in \mathbb{H}$ is explicit, defined as follows.

- $\alpha \in \Pi$, $r_\alpha := t_{s_\alpha} - \frac{k_\alpha}{\alpha}$,

- $w = s_1 \cdot \cdots \cdot s_k$, $R_w := \prod r_{\alpha_i}$.

R_w does not depend on the particular minimal decomposition of w into simple reflections. Its main property is that $\textcolor{red}{R_w \omega = w^{-1}(\omega) R_w}$.

One would like to relate the form for $\star$ with that for $\bullet$ with the expectation that $\bullet$ is easier. We need the classification of hermitian modules.

Tempered modules are unitary for $\star$, because they come from L^2 of a group (some k_α come from work of Opdam).

$\textcolor{blue}{\text{An essential property of } A_{w^0} \text{ is that it is analytic for } \sigma_0 \text{ tempered}}$
$\textcolor{blue}{\text{and } \nu \text{ satisfying (i) of the theorem.}}$

$\textcolor{red}{\text{We will use } r_\alpha := t_\alpha \alpha - k_\alpha \text{ in some later formulas.}}$

Tempered modules and ●

Recall a defined by $a(\omega) = -w_0\omega$ and $a(t_w) = t_{w^{-1}}$. If (σ, U) is tempered, so is $(\sigma \circ a, U)$. We restrict attention to Hecke algebras of geometric type.

a) $\sigma \cong \sigma \circ a$. Tempered representations are parametrized by $G-$conjugacy classes of pairs $\{e, \psi\}$ where $e \in \mathfrak{g}$ is a nilpotent element, and ψ a character of the component group $A(e)$ of the centralizer of e of generalized Springer type, matching it with a W-representation σ_ψ. The Jacobson-Morozov theorem implies that a stabilizes the class of e. The claim follows from the fact that a is the identity on $\widehat{W}$.

b) The intertwining operator $\theta : U \longrightarrow U$ corresponding to a is unique up to a constant, satisfies $\theta^2 = Id$, and so it can be normalized to satisfy $\theta^* = \theta$.

Relation between $\star$ and $\bullet$

Extend $\mathbb{H}$ by a so that $at_w = t_{w_0 w w_0} a$ and $a\omega = (-w_0 \omega)a$. The star operations are extended by $a^\bullet = a$, $a^* = a$.

Let π be a representation of the extended algebra. Suppose a module (π, U) has a $\bullet$−hermitian form $\langle\ ,\ \rangle_\bullet$. We can define

$$(v_1, v_2)_\star := \langle \pi(at_{w_0})v_1, v_2 \rangle_\bullet.$$

Keeping in mind that $at_{w_0} = t_{w_0} a$, and $(at_{w_0})t_w = t_w(at_{w_0})$,

$$(\pi(t_w)v_1, v_2)_\star = \langle \pi(at_{w_0})\pi(t_w)v_1, v_2 \rangle_\bullet = (v_1, \pi(t_w^\star)v_2)_\star$$

$$(\pi(a)v_1, v_2)_\star = (v_1, \pi(a^*)v_2)_\star$$

$$(\pi(\omega)v_1, v_2)_\star = (v_1, \pi(\omega^*)v_2)_\star$$

Induced Hermitian Modules

- (σ, U_σ) a representation of $\mathbb{H}_M$, $\operatorname{Ind}(M, \sigma) := \mathbb{H} \otimes_{\mathbb{H}_M} U$ with action $\pi(h)h_1 \otimes v := hh_1 \otimes v$.

- $(\sigma^{\bullet_M}, U^h)$ and $(\sigma^{\star_M}, U^h)$ the representations on the hermitian dual space U^h.

- $(\pi^\bullet, \operatorname{Ind}(M, \sigma)^h)$ and $(\pi^\star, \operatorname{Ind}(M, \sigma)^h)$ the representations on the hermitian dual.

- The space $\operatorname{Ind}(M, \sigma)^h$ can be identified with $\operatorname{Hom}_{\mathbb{H}_M}[\mathbb{H}, \mathbb{C}] \otimes U^h$ so a typical element is $\{t_x^h \otimes v^h\}$ where $x \in W/W(M)$ and $v^h \in U^h$.

Theorem. *[B-Ciubotaru] The map*

$$\Phi(t_x^h \otimes v^h) := t_{xw^0 a_M} \otimes a v^h$$

is an $\mathbb{H}-$equivariant isomorphism between $\left(\pi_\sigma^\bullet, X(M,\sigma)^h\right)$ and $\left(\pi_\sigma, X(aM, a\sigma^h)\right)$ where the action on $a\sigma^h$ is given by $\bullet a_{(M)}$.

Similarly for $\star$, but the relation between $\star_G$ and $\star_M$ is more complicated.

Example. $\Pi_M = \emptyset$. *The standard module is $X(\nu)$ the full principal series.*

$$(\pi^\bullet, X(\nu)^h) \cong (\pi, X(w_0 \overline{\nu}))$$

$$(\pi^\star, X(\nu)^h) \cong (\pi, X(-\overline{\nu}))$$

This makes it precise which irreducible modules are hermitian.

For $\bullet$ you need $w_0 \overline{\nu}$ to be in the same Weyl orbit as ν, same as ν and $\overline{\nu}$ must be in the same Weyl orbit.

For $\star$ you need $-\overline{\nu}$ to be in the same Weyl orbit as ν.

Corollary. *$L(M, \sigma_0, \nu)$ admits a nondegenerate hermitian form for*

●: *any $\nu \in V_M^\vee$,*

★: *if and only if there exists $w \in W$ such that $w\nu = -\nu$, and $w \circ \sigma_0 \cong \sigma_0$ (in this case $aM = M$).*

We restrict to the case ν real and $w_0 \nu = -\nu$.

Remark: It is possible to dispense with $w_0 \nu = -\nu$ by considering the algebra extended by a as in [BC1]. We will not do so in this talk.

The relation between ★ and ● is essentially that between the signature of a hermitian matrix A and another $T_0 A$ which is also hermitian. This is a simple relation, but rather complicated to make explicit.

Sesquilinear Forms

A $\bullet-$invariant sesquilinear form on $\operatorname{Ind}(M, \sigma)$ is equivalent to defining an $\mathbb{H}$-equivariant map

$$\mathcal{I} : (\pi, \operatorname{Ind}(M, \sigma)) \longrightarrow (\pi^\bullet, \operatorname{Ind}(M, \sigma)^h).$$

We call $\mathcal{I}$ hermitian if $\mathcal{I}^h = \mathcal{I}$ or equivalently $\mathcal{I}(v)(w) = \overline{\mathcal{I}(w)(v)}$, for all $v, w \in X(M, \sigma)$.

For the case $X(M, \sigma_0, \nu)$ and $L(M, \sigma_0, \nu)$ one can write down a formula for the hermitian form. It depends on the structure of σ_0 which can be highly nontrivial.

Spherical Principal Series

Assume from now on that ν is real. For ν regular,

- $A := S(V_{\mathbb{C}})$

- $X(\nu) := \mathbb{H} \otimes_A \mathbb{C}_\nu$.

- $\langle h_1, h_2 \rangle_{\bullet, \nu} = \epsilon_A \left(t_{w_0} h_2^\bullet h_1 R_{w_0} \right) (w_0 \nu)$.

- $\langle h_1, h_2 \rangle_{\star, \nu} = \epsilon_A \left(h_2^\star h_1 R_{w_0} \right) (w_0 \nu)$.

Any element $h \in \mathbb{H}$ can be written uniquely as $h = \sum t_w a_w$ with $a_w \in S(V_{\mathbb{C}})$. Then $\epsilon_A(h) := a_1$.

For ν singular, assume it is dominant, and let M be the Levi component for which ν is central. Let t^0 be the shortest element in the coset $t_{w_0} W(M)$, and R^0 the corresponding element.

- $\langle h_1, h_2 \rangle_{\bullet, \nu} = \epsilon_A \left(t_{w^0} h_2^{\bullet} h_1 R^0 \right) (w_0 \nu)$.

- $\langle h_1, h_2 \rangle_{\star, \nu} = \epsilon_A \left(h_2^{\star} h_1 R^0 \right) (w_0 \nu)$.

The quotient by the radical is the irreducible spherical representation corresponding to ν.

Write $R^0 = \sum t_w a_w$ with $a_w \in S(V)$. Then

- $\langle t_x, t_y \rangle_{\bullet, \nu} = a_{x^{-1} y (w^0)^{-1}} (w_0 \nu)$.

- $\langle t_x, t_y \rangle_{\star, \nu} = a_{x^{-1} y} (w_0 \nu)$.

These formulas give the matrices of the hermitian forms.

They generalize to arbitrary Langlands parameters.

Example

Recall the elements R_x, and assume $a = Id$. Then

$$R_x^\bullet = (-1)^{\ell(x)} R_{x^{-1}}$$

$$R_x^\star = (-1)^{\ell(x)} t_{w_0} R_{x^{-1}} t_{w_0},$$

If $\alpha(\nu) > 0$ for all $\alpha \in \Pi$, then $R_x \otimes \mathbb{C}_\nu$ is a basis of $X(\nu)$ formed of eigenvectors for V.

$$\langle R_x, R_y \rangle_\bullet = \begin{cases} 0 & \text{if } x \neq y, \\ (-1)^{\ell(x)} \prod_{x^{-1}\alpha < 0} (1 - \alpha^2)(w_0 \nu) & \text{if } x = y \end{cases}$$

This formula computes the *Jantzen filtration* of $X(\nu)$ explicitly.

$\langle R_x, R_y \rangle_\star$ is more complicated.

This illustrates how $\bullet$ can be much simpler than $\star$.

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One Consequence

To determine whether an irreducible module is unitary, it has to be hermitian, and the $*-$form must be positive definite. This is the same as determining that the form on the standard module is positive semidefinite. The standard module inherits a filtration such that every successive quotient has a nondegenerate invariant form. The form changes with respect to the continuous parameter ν in a predictable way, and one can talk about a signature. The problem is that a given irreducible module can have two nondegenerate forms up to a positive constant.

Problem: Need to keep track of this ambiguity.

The next result addresses this issue.

Consider again the case of a Hecke algebra of geometric type. The classification results of Kazhdan-Lusztig imply that standard modules have lowest $W-$types.

In the case of $Re\langle\nu,\alpha\rangle > 0$ they determine the Langlands quotient $L(M,\sigma_0,\nu)$ (the Langlands quotient is the unique irreducible subquotient containing **all** the lowest $W-$types with full multiplicity occuring in $X(M,\sigma_0,\nu)$).

These lowest $W-$types can be used to single out one of the forms. Then one can look for an explicit algorithm to write out the singature of a module.

Proposition (B-Ciubotaru). *When $L(M, \sigma_0, \nu)$ is hermitian, the nondegenerate form can be normalized so that:*

the $\bullet$-form is positive on the lowest W-types.

More precisely, suppose $a = Id$ so that t_{w_0} is central. Let $\deg(\mu)$ be the lowest degree so that μ occurs in the harmonics of $S(V_{\mathbb{C}})$. The $\star$-form on a lowest W-type μ is given by $(-1)^{\deg(\mu)}$.

The general formula is a little more complicated. The signature on a lowest W-type μ is given by the trace of t_{w_0}.

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In the real case, tensoring with finite dimensional representations plays an essential role in determining filtrations and hermitian forms. This is not available in the case of an affine graded Hecke algebra. There are results of Lusztig and Ginzburg, and work/ideas of Grojnowski aimed at computing composition factors of standard modules. The facts about filtrations of standard modules that are necessary for a treatment parallel to the real case are only conjectural (as far as I know).

Thank You,

and

HAPPY BIRTHDAY TO WILFRIED!

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Complementary Series

continuation ...

B_n with parameter $k_\alpha = 1$ for α long, $k_\alpha = c > 0$ for α short, spherical generic parameter, ν real:

$$0 \le \nu_1 \le \cdots \le \nu_n, \quad \text{no } \nu_i = c, \quad \pm\nu_i \pm \nu_j = 1.$$

The next slide gives the generic spherical complementary series. The parameters are more general than the geometric ones studied by Lusztig. Similar results hold for the other types.

Theorem. *The complementary series for type B is*

$0 < c \leq 1/2 : 0 \leq \nu_1, \ldots, \nu_1 < \cdots < \nu_k, \ldots, \nu_k < c.$

$1/2 < c \leq 1 :$

$0 \leq \nu_1 \leq \cdots \leq \nu_k \leq 1/2 < \nu_{k+1} < \nu_{k+2} < \cdots < \nu_{k+l} < c$

so that $\nu_i + \nu_j \neq 1$ for $i \neq j$ and there are an even number of ν_i such that $1 - \nu_{k+1} < \nu_i < c$ and an odd number of ν_i such that $1 - \nu_{k+j+1} < \nu_i < 1 - \nu_{k+j}$.

$1 < c :$ *(joint with D. Ciubotaru)*

1. *$0 \leq \nu_1 \leq \nu_2 \leq \cdots \leq \nu_m < c$ satisfy the unitarity conditions for the case $1/2 < c \leq 1$.*

2. *$\nu_{j+1} - \nu_j > 1$ for all $j \geq m + 1$.*

3. *either $\nu_{m+1} - \nu_m > 1$ or, if $1 - \nu_{k+1} < \nu_m < 1 - \nu_k$ ($k + m$ is necessarily odd), then $1 + \nu_l < \nu_{m+1} < 1 + \nu_{l+1}$, with $k \geq l + 1$ and $m + l$ even.*

In terms of hyperplane arrangements, the regions of unitarity are precisely those which are adjacent to a wall for which the parameters are unitary for a Levi component.

A region where the parameter is not unitary has a wall where a composition factor has $W-$types

$(1, n-1) \times (0)$ and $(n-1) \times (1)$ of opposite sign.

Originally we proved the theorem by using explicit formulas for the intertwining operator on these $W-$types. Using the $\bullet-$form and the R_x we can reduce to a (simpler) combinatorial argument about the Weyl group. We expect this to be useful for the nonspherical case.