A problem in Euclidean Geometry

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I describe below an elementary problem in Euclidean (or Hyperbolic) geometry which remains unsolved more than 10 years after it was first formulated. There is a proof for $n = 3$ and (when the ball is the whole of 3-space) when $n = 4$. There is strong numerical evidence for $n \leqslant 30$.

Let $(x_1, x_2, ...x_n)$ be n distinct points inside the ball of radius R in Euclidean 3-space. Let the oriented line $x_i x_j$ meet the boundary 2-sphere in a point t_{ij} (regarded as a point of the complex Riemann sphere $(C \cup \infty)$). Form the complex polynomial p_i, of degree $n-1$, whose roots are t_{ij} : this is determined up to a scalar factor. The open problem is

Conjecture 1 For all $(x_1, ..., x_n)$ the n polynomials p_i are linearly independent.

Conjecture 1 is equivalent to the non-vanishing of the determinant D of the matrix of coefficients of the p_i. In fact there is a natural way of normalizing this determinant (independent of the choice of scalar factors) so that D becomes a continuous function of $(x_1, ..., x_n)$ which is $SL(2, C)$ - invariant (using the ball model of hyperbolic 3-space) Conjecture 1 can now be refined to

Conjecture 2 $|D| \geq 1$ with equality only for collinear points.

There are other versions of this conjecture, of which the most appealing and general involves 2 ellipsoids S, S' in 3-space with S inside S'. Replacing the 2-sphere above by an ellipsoid and, taking a sequence of points x_i inside S, we get two determinants D, D'. The third conjecture (which implies Conjecture 2) is

Conjecture 3 $|D'| > |D|$

More details and background can be found in the references below (but Conjecture 3 is new).

References

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