Feb 7. 2006 1:00PM – 2:30PM Tuesday Kevin Buzzard 2nd 1

Hecke ops & Diamond ops.

are really quite general things

big space M + action of big gp Δ

$\Gamma \subseteq \Delta$. M^Γ is fixed points

If $\Gamma_1 \subseteq \Gamma_2$ then inclusion $M^{\Gamma_2} \subseteq M^{\Gamma_1}$

If $g \in \Delta$, then $f \mapsto g * f$ induces a map

$$M^\Gamma \longrightarrow M^{g\Gamma g^{-1}}$$

Finally if $\Gamma_1 \subseteq \Gamma_2$ finite index

& $\Gamma_2 = \coprod_{i=1}^{n} \gamma_i \Gamma_1$, then $f \longmapsto \sum \gamma_i * f$

is a map $M^{\Gamma_1} \longrightarrow M^{\Gamma_2}$

Hecke operators. explicit instances of these ideas

<u>Example of how Hecke ops act on q-expansions</u>

If $f \in M_k(\Gamma_1(N))$

& if $\langle d \rangle f = \chi(d) f$ for some Dirichlet char.

then If I know q-exp of f, I can compute q-exp of $T_p f$

$f = \sum a_n q^n$

If $p \mid N$. $T_p(f) = \sum a_{np} q^n$ $\qquad \Gamma_0(N)/\Gamma_1(N) = (\mathbb{Z}/N)^\times$

& if $p \nmid N$. $T_p f = \sum a_{np} q^n + p^{k-1} \chi(p) \sum a_n q^{np}$

One key consequence: if $f = q + \quad = \sum a_n q^n$ is an eigenform for

all the T_p. then one checks that a_p is the eigenvalue of T_p

(a_n is the eigenvalue of T_n.

In fact one can <u>use this to explain</u>

why many common modular form have q-expansions in $\mathbb{Q}[[q]]$

General theorem $\Rightarrow$ eigenvalues of T_n are algebraic numbers.

So far, $X_1(N)$ etc are compact Riemann surface

Exercise: $\Gamma_1(N) \backslash \mathfrak{h}$ naturally bijects with pairs (E, P) $\quad E$: elliptic curve$/\mathbb{C}$, $P \in E$ pt of order N

Dictionary $\tau \in \mathfrak{h} \longrightarrow \left(\mathbb{C}/_{\mathbb{Z}\oplus\mathbb{Z}\tau}, \frac{1}{N} \right)$

Amazing fact: this simple idea (E) can be translated into alg. geo.
& in particular it "makes sense" over an arbitrary field of char 0

$X_0(2)_{\mathbb{Q}}$ | — concrete description

Facts: If $\Gamma \subseteq SL_2(\mathbb{Z})$ is a congruence subgp & Γ has no elliptic elements, then there an alg. curve $Y(\Gamma)$ over $\mathbb{Q}$ parametrizing elliptic curves (over $\mathbb{Q}$-schemes) with a "level Γ structure"

e.g. if $\Gamma = \Gamma_1(N)$, $N \geq 4$

then "level Γ structure" can mean "point of order N" (or "embedding of μ_N")

More precisely, there's an alg. curve $Y_1(N)/\mathbb{Q}$
& an elliptic curve

$$\begin{array}{c} E \\ \downarrow \\ Y_1(N) \end{array}$$

equipped with a "point $P: Y_1(N) \to E$ of order N"

s.t. if S is any scheme over Spec $\mathbb{Q}$ & A/S is any elliptic curve with a point $Q \in A(S)$ of order N.
then there's a unique map

$S \longrightarrow Y_1(N)$ s.t. pull back of $\frac{E}{P}$ is $\frac{A}{Q}$

In particular, if $K \supseteq \mathbb{Q}$ is any field
$Y_1(N)(K)$ bijects naturally with iso classes of pairs (A, Q)

A : ell. curve/K
$Q \in A(K)$ pt of order N

There are simple construction of $Y_1(N)$ & fancy ones too.

Similarly $Y(N)$, for $N \geq 3$, is an alg. curve/$\mathbb{Q}$

parametrizing pairs $(E, \alpha: \mathbb{Z}/_{N\mathbb{Z}} \times \mu_N \xrightarrow{\sim} E[N])$
α an isom preserving natural pairing.

Algebro geometric $Y(N)$ & $Y_1(N)$
have associated Riemann Surfaces iso to old $Y(N)$, $Y_1(N)$.

What about $Y_1(N)$. N small. 9

& What about $Y_0(N)$, trying to parametrize pairs (E, C) C cyclic gp of order N

These things also exist in alg. geom. because in alg. geo. you can sometimes quotient cat by a finite gp. & you can fine

General construction of $Y_0(N)$: choose odd prime p. $p \nmid N$

The moduli problem representing E, C, φ ← full level p-str.

is representable by a curve ↑ cyclic gp of order N.

& the curve is affine $\operatorname{Spec} A$

& it has a natural action of $SL_2(\mathbb{Z}/p\mathbb{Z})$

Invariants $A^{SL_2(\mathbb{Z}/p\mathbb{Z})}$, take Spec of this,

Similar trick does & its $Y_0(N)$ $Y_1(N)$, $N<4$

These curves $Y_0(N)$ & $Y_1(N)$ $N<4$ aren't quite the solutions to natural moduli problems involving elliptic curves but there's still a canonical bijection of sets

$Y_0(N)(K) =$ iso classes of pairs E, C $C \subset E$ cyclic of order N

if K is alg. closed

If $N \geq 4$ then there's a sheaf ω on $Y_1(N)$, whose analytification is old ω of last time

$$E \xrightarrow{\pi} Y_1(N)$$

$\omega = \pi_* \Omega^1_{E/Y_1(N)}$

ω is an invertible sheaf on $Y_1(N)$

& fiber of ω at a pt $x \in Y_1(N)$ is just $H^0(E, \Omega^1)$

$\updownarrow$

(E, p)

However "quotient trick" doesn't work on ω,

& $Y_0(N)/\mathbb{Q}$ etc don't have a natural ω

[although they may have a natural power of ω, eg. $Y_0(1)$ has a sheaf on it

Curves $Y_0(N)$ $Y_1(N)$ etc have natural compactifications

$X_0(N)$, $X_1(N)$

ω extends if no irregular cusps

Can think of this all over $\mathbb{Q}$

Can define Hecke ops as endomorphs of $\underbrace{H^0(X_1(N), \omega^{\otimes k})}_{\text{f.d. } \mathbb{Q}\text{-v.sp}}$

then $\otimes \mathbb{C}$ recover old spaces.

One last remark:

If $X_1(N)$ is the modular curve $/\mathbb{Q}$

$H^0(X_1(N), \omega^{\otimes k})$ is a $\mathbb{Q}$-lattice in the $M_k(\Gamma_1(N))$ of last time

$\quad \| $

$M_k(\Gamma_1(N), \mathbb{Q})$

Basic fact: If $f \in M_k(\Gamma_1(N))$, then $f \in M_k(\Gamma_1(N), \mathbb{Q})$

$\Leftrightarrow$ q-expansion in $\mathbb{Q}[[q]]$

[Remark: for this to be really true,

I should define $X_1(N)$ as parametrizing (E, β), $\beta: \mu_N \hookrightarrow E)$

Katz "p-adic properties of modular schemes & modular forms".

In particular, $G_k = \frac{\zeta(1-k)}{2} + \sum \sigma_{k-1}(n) \cdot q^n \in M_k(\Gamma_1(N), \mathbb{Q})$

2 strands to this course

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(1) p-adic theory - Eigencurves - in particular "writing eigencurve down"

$\uparrow$

Modular form have got good "p-adic analytic" properties,

Example: one can check that for p: prime,

if $S = \{ k \in \mathbb{Z},\ k \geq 2 \ \&\ k \equiv 0 \bmod \begin{matrix} p-1 \\ 2 \end{matrix} \quad \begin{matrix} p \text{ odd} \\ p \text{ even} \end{matrix} \}$

& $k \in S$

then the power series

$$E_k = 1 + \frac{2}{(1-p^{k-1})\zeta(1-k)} \cdot \sum_{n \geq 1} \sigma^*_{k-1}(n) q^n$$

$E_k \equiv 1 \pmod p$

$k \in S$

$\sigma^*_t(n) = \sum_{\substack{0<d \mid n \\ p \nmid d}} d^t$

① is the q-expansion of a modular form wt k, level p

② varies p-adically continuously with $k \in S$

— one can check (lemma 7.12 & remarks after lemma 7.9 Washington "Cyclotomic field")

that q-expansion of E_k is in $\mathbb{Z}_p[[q]]$

& if $k \equiv k' \bmod p^N$, $k, k' \in S$ then $E_k \equiv E_{k'} \bmod p^N$

Hence Eisenstein series seem to move p-adically continuously

Does $M_k(\Gamma_1(N); \mathbb{Q})$ move p-adically continuously?

No. if $k \equiv k' \bmod p^N$

but $k' \gg k$

then $M_{k'} \gg M_k$

$k \equiv k' \equiv 0 \bmod p-1$
$k \equiv k' \bmod p^n$
$\Rightarrow$
$p \nmid d$, $d^k \equiv d^{k'} \bmod p^n$

Coleman saw how to expand $M_k(\Gamma_1(N))$ so that was ∞-dim'l V_k,

In fact. & bigger ~~M_k~~ M_k's dim'l vary continuously

$S \subseteq W^\circ$ = open p-adic disc

& Coleman defined "overconvergent modular forms of wt k" for any $k \in W^\circ$.

② p-adic Langlands for GL_2

Let $f \in S_k(\Gamma_1(N))$ be an eigenform.

$f = q + \cdots$

coeff. of f generate a number field E

If λ is a prime of E, $\lambda | \ell$

then Deligne ($k \geq 3$)
Eichler Shimura ($k = 2$)
Deligne-Serre ($k = 1$)

associates to f a Galois rep'n

$\rho_f : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(E_\lambda)$ satisfying certain properties.

f: level N.: $N \mid \ell$

e.g. if p is prime, $p \nmid N\ell$, then ρ_f is unr. at p

& $\rho_f(\mathrm{Frob}_p)$ has char. poly. $X^2 - a_p X + p^{k-1} \cdot \chi(p)$

Frob_p is arithmetic Frobenius

Cyclotomic char has Hodge-Tate (1) -1

$\begin{cases} \chi : \text{char of } f \\ a_p \text{ is eigen-value of } T_p \end{cases}$

& ρ_f is cont. odd irreducible

ρ_f has Hodge-Tate weights 0 & $k-1$.

Q) What is $\rho_f|_{D_p}$ for the other primes p?

$\ell \nmid N$ Here's the answer for $p \neq \ell$: Associated to f is an irred infinite dimensional rep'n Π_f of $GL_2(\mathbb{A}^\infty)$

$\hat{\mathbb{Z}} \otimes \mathbb{Q}$

This is not a mystery. Consider $f|_k \gamma$, $\forall \gamma \in GL_2^+(\mathbb{Q})$

& one checks $\Pi_f = \bigotimes_p' \Pi_{f,p}$

irreducible rep'n of $GL_2(\mathbb{Q}_p)$

e.g. if $p \nmid N$, $\Pi_{f,p}$ is "unramified principal series"

& is determined by a_p & k & $\chi(p)$

Local Langlands conj for GL_2 $\leftarrow$ a theorem

Associates to $\Pi_{f,p}$ as above

$\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\hookrightarrow \overline{\mathbb{Q}}_p$ a 2-dim'l rep'n of the Weil-Deligne gp of $\mathbb{Q}_p$

& then an elementary construction gives a 2-dim rep

$(\rho_{f,p} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \longrightarrow GL_2(\overline{\mathbb{Q}}_\ell))$ $p \neq \ell$

"Local-global compatibilities" (thm of Carayol)

$\rho_{f,p} \cong$ restriction of ρ_f to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$

In particular, our former statement about $\rho_f|_{G_{\mathbb{Q}_p}}$ for $p \nmid N\ell$ is a conseq. of this

$$\begin{array}{ccc} f & \longrightarrow & \rho_f \\ \downarrow & \circlearrowleft & \downarrow \\ \Pi & & \rho_f|D_p \\ & \searrow \Pi_p \nearrow & \end{array}$$

Rk: One can almost prove Local Langlands for GL_2 by "writing down both sides"

What about $p = \ell$?

$$\begin{tikzcd} f \arrow[r, mapsto] \arrow[d] & \rho_f \arrow[r] & \rho_f|_{D_p} \\ \Pi_f \arrow[dr] & & \\ & \Pi_{f,p} & \end{tikzcd}$$

$Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to GL_2(\overline{\mathbb{Q}})$ These are lots of these

→ He will write down element!

$\Pi_{f,p}$ = not that many of these

$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

P. Colmez

Q1) Can we recover $\Pi_{f,p}$ from $\rho_f|_{D_p}$?

Yes (T. Saito)

Q2) Can we go other way? Answer must be "no".

– So what can we do?

(I)
– use global nature of f to put a little extra str. on $\Pi_{f,p}$

(II) Prove p-adic Local Langlands for GL_2