

Hecke ops & Diamond ops.

are really quite general things

big space M + action of big \mathbb{Z}^d

$\Gamma \trianglelefteq \Delta$, M^Γ is fixed points

If $\Gamma_1 \subseteq \Gamma_2$ then inclusion $M^{\Gamma_2} \subseteq M^{\Gamma_1}$

If $g \in \Delta$, then $f \mapsto g * f$ induces a map

$$M^\Gamma \rightarrow M^{g\Gamma g^{-1}}$$

Finally if $\Gamma_1 \subseteq \Gamma_2$ finite index

$$\& \Gamma_2 = \coprod_{i=1}^n \gamma_i \Gamma_1$$

then $f \mapsto \sum \gamma_i * f$

is a map $M^{\Gamma_2} \rightarrow M^{\Gamma_1}$

Hecke operators. explicit instances of these ideas

Example of how Hecke ops act on q -expansions

If $f \in M_k(\Gamma_1(N))$

& if $\langle d \rangle f = \chi(d) f$ for some Dirichlet char.

then if I know q -exp of f , I can compute q -exp of $T_p f$

$$f = \sum a_n q^n$$

$$\text{if } p \mid N, T_p(f) = \sum a_{np} q^n$$

$$\rho_0(N) \times \rho_1(N) = \left(\frac{\mathbb{Z}}{N}\right)^2$$

$$\& \text{if } p \nmid N, T_p f = \sum a_{np} q^n + p^{k-1} \chi(p) \sum a_n q^{np}$$

One key consequence: if $f = \sum a_n q^n$ is an eigenform for

all the T_p , then one checks that a_p is the eigenvalue of T_p
 (a_n is the eigenvalue of T_n)

In fact one can use this to explain

why many common modular forms have q -expansions in $\mathbb{Q}(\mu_N)$.

General theorem \Rightarrow eigenvalues of T_n are algebraic numbers.

So f_{new} , $X_1(N)$ etc are compact Riemann surface

Exercise:

$$\Gamma_0(N) \backslash \mathbb{H}$$

naturally bijects with pair (E, P)

E : elliptic curve/ \mathbb{C}
 $P \in E$ pt of order N

Dictionary $\tau \in \mathbb{H} \rightarrow (\mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}\tau i, \frac{1}{N})$

Amazing fact: this simple idea can be translated into alg. geo.

& in particular, it "makes sense" over an arbitrary field of

$X_0(N)_{\mathbb{Q}}$ — concrete description

char 0

Facts: If $\Gamma \subseteq SL_2(\mathbb{Z})$ is a congruence subgroup & Γ has no elliptic elements, then there an alg. curve $Y(\Gamma)$ over \mathbb{Q} parametrizing elliptic curves (over \mathbb{Q} -schemes) with a "level Γ structure"

e.g. if $\Gamma = \Gamma_1(N), N \geq 4$

then "level Γ structure" can mean "point of order N "

More precisely, there's an alg. curve $Y_1(N)/\mathbb{Q}$ (on "embedding of $\mathbb{C}/N\mathbb{Z}$ ")

& an elliptic curve E equipped with a point $P: Y_1(N) \rightarrow E$ of order N

s.t. if S is any scheme over $\text{Spec } \mathbb{Q}$ & A/S is any elliptic curve with a point $Q \in A(S)$ of order N .

then there's a unique map

$$S \rightarrow Y_1(N) \quad \text{st } \begin{array}{c} \text{pullback of } E \text{ is } A \\ P \quad Q \end{array}$$

In particular, if $K \supseteq \mathbb{Q}$ is any field

$Y_1(N)(K)$ bijects naturally with iso classes of pairs (A, Q)

A : elliptic curve/ K
 $Q \in A(K)$ pt of order N

There are simple construction of $Y_1(N)$ & fancy ones too.

Similarly $Y(N)$, for $N \geq 3$, is an alg. curve/ \mathbb{Q} .

parametrizing pairs $(E, d: \mathbb{Z}/N\mathbb{Z} \times \mu_N \xrightarrow{\sim} E[N])$
 of an isom preserving natural pairing.
 Algebraic geometry $Y(N)$ & $Y_1(N)$
 have associated Riemann Surfaces iso. to old $Y(N), Y_1(N)$.

What about $Y_1(N)$, N small.

& what about $Y_0(N)$, trying to parametrize pairs

(E, C) cyclic gp of order N

These things also exist in alg. geom. because in alg. geom. you can sometimes quotient out by a finite gp. & you can find

General construction of $Y_0(N)$: choose odd prime p , $p \nmid N$

The moduli problem representing E, C is representable by a curve & the curve is affine $\text{Spec } A$

full level p -str. cyclic gp of order N .

& it has a natural action of $\text{SL}_2(\mathbb{Z}/p\mathbb{Z})$

Invariants $A^{\text{SL}_2(\mathbb{Z}/p\mathbb{Z})}$, take Spec of this,

Similar trick does $Y_1(N)$, $N \nmid 4$

These curves $Y_0(N)$ & $Y_1(N)$ aren't quite the solutions to natural moduli problems involving elliptic curves, but there's still a canonical bijection of sets

$$Y_0(N)(k) = \text{iso classes of pairs } (E, C)$$

(E, C) cyclic of order N

If $N \nmid 4$, then there's a sheaf ω on $Y_1(N)$, whose analytification is old ω of last time

$$\begin{array}{c} E \\ \downarrow \pi \\ Y_1(N) \end{array}$$

$$\omega = \pi_* \Omega^1_{E/X_1(N)}$$

ω is an invertible sheaf on $Y_1(N)$

& fiber of ω at a pt $x \in Y_1(N)$ is just

$$\begin{array}{c} \downarrow \\ (E, p) \quad H^0(E, \Omega^1) \end{array}$$

However "quotient trick" doesn't work on ω ,

& $Y_0(N)/\mathbb{Q}$ etc don't have a natural ω

[although they may have a natural power of ω e.g. $Y_0(1)$ has a sheaf on it

Curves $Y_0(N)$, $X(N)$ etc have natural compatibilities

$$X_0(N) : X_1(N)$$

ω extends if no irregular cusps

Can think of this all over \mathbb{Q}

Can define Hecke ops as endomorphisms of $H^0(X_1(N), \omega^{\otimes k})$

One last remark:

IF $X_1(N)$ is the modular curve / \mathbb{Q}

$H^0(X_1(N), \omega^{\otimes k})$ is a \mathbb{Q} -lattice in the $M_k(\Gamma_1(N))$ of last time
 $M_k(\Gamma_1(N), \mathbb{Q})$

f.d. \mathbb{Q} -v. sp
 $\omega \otimes \mathbb{Q}$ recovers
 old space.

Basic fact: IF $f \in M_k(\Gamma_1(N))$, then $f \in M_k(\Gamma_1(N), \mathbb{Q})$

[Remark: for this to be really true, \Leftrightarrow q -expansion in $\mathbb{Q}[[q]]$

I should define $X(N)$ as parametrizing (E, β) , $\beta: \mathbb{Z}/N\mathbb{Z} \rightarrow E$

Katz "p-adic properties of modular schemes & modular forms".

In particular, $E_k = \frac{\zeta(1-k)}{2} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \in M_k(\Gamma_1(N), \mathbb{Q})$

\mathbb{Q} strands to this course

① p-adic theory - Eigencurves. - in particular "writing eigencurve down"

Modular forms have got good "p-adic analytic" properties

Example: one can check that for p : prime,

if $S = \left\{ \begin{array}{l} k \in \mathbb{Z}, k \geq 2 \text{ \& } k \equiv 0 \pmod{p-1} \\ \& k \in S \end{array} \right. \left. \begin{array}{l} p \text{ odd} \\ p \text{ even} \end{array} \right\}$

then the power series

$$E_k = 1 + \frac{2}{(1-p^{k-1})S(1-k)} \sum_{n \geq 1} \sigma_{k-1}^*(n) q^n$$

$$\sigma_k^*(n) = \sum_{d|n} d^k$$

$$E_k \equiv 1 \pmod{p} \quad k \in S$$

① is the q -expansion of a modular form of k level p

② varies p -adically continuously with $k \in S$

— one can check (Coleman 2.12 & remarks after lemma 7.9 Washington "Galois Fields")
that q -expansion of E_k is in $\mathbb{Z}_p[[q]]$

& if $k \equiv k' \pmod{p^N}$ then $E_k \equiv E_{k'} \pmod{p^N}$
 $k, k' \in S$

Hence Eisenstein series seem to move p -adically continuously

Does $M_k(\Gamma_1(N); \mathbb{Q})$ move p -adically

No, is $k \equiv k' \pmod{p^N}$ continuously?
but $k' \gg k$

$$\begin{aligned} k \equiv k' \equiv 0 \pmod{p^1} \\ k \equiv k' \pmod{p^N} \\ \Rightarrow p \mid d, d^k = d^{k'} \pmod{p^N} \end{aligned}$$

then $M_{k'} \gg M_k$

Coleman saw how to expand $M_k(\Gamma_1(N))$ so that was ∞ -dim V_k

In fact, & bigger $M_{k'}^i$ did vary continuously
 $M_{k'}^i$

$S \subseteq W^\circ =$ open p -adic disc

& Coleman defined "overconvergent modular forms of wt k "
for any $k \in W^\circ$

② p -adic Langlands for GL_2

Let $f \in \mathcal{O}_k(\Gamma_1(N))$ be an eigenform.

$$f = f + \dots$$

coeff. of f generate a number field E

If λ is a prime of E , A.I.L.

then Deligne ($k \geq 3$)

Eichler Shimura ($k=2$)

Deligne-Serre ($k=1$)

associates to f a Galois rep'n

$$\rho_f : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(E_\lambda) \text{ satisfying certain properties.}$$

f : level N : $2|N$

e.g. if p is prime, $p \nmid N$, then ρ_f is unramified at p

& $\rho_f(\text{Frob}_p)$ has char. poly. $X^2 - a_p X + p \cdot \chi(p)$

Cyclotomic
char has
Hodge-Tate ± 1

Frob_p is arithmetic Frobenius

χ : char of f

a_p is eigen-value of T_p

& ρ_f is cont. add. irreducible

ρ_f has Hodge-Tate weights 0 & $k-1$.

Q)

What is $\rho_f|_D$ for the other primes p ?

Q+N)

Here's the answer for $p \neq l$: Associated to f is an infinite dimensional rep'n Π_f of $GL_2(\mathbb{A}^\infty)$

This is not a mystery. Consider $f|_K$, $V_{\chi \otimes \mathbb{Q}} \in GL_2^+(\mathbb{Q})$

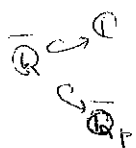
& one checks $\Pi_f = \bigotimes_p \Pi_{f,p}$

e.g. if $p \nmid N$, $\Pi_{f,p}$ is "unramified principal series" irreducible rep'n of $GL_2(\mathbb{Q}_p)$

& is determined by a_p & k & $\chi(p)$

Local Langlands conj for $GL_2 \leftarrow$ a theorem

Associates to $\Pi_{f,p}$ as above.



a 2-dim rep'n of the Weil-Delegue gp of $\bar{\mathbb{Q}}_p$

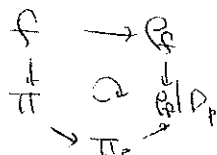
& then an elementary construction gives a 2-dim rep

$$(\rho_{f,p} : \text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \rightarrow GL_2(\bar{\mathbb{Q}}_p))$$

"Local-global compatibilities" (thm of Carayal) $\left. \begin{array}{l} \\ \end{array} \right\} p \neq l$

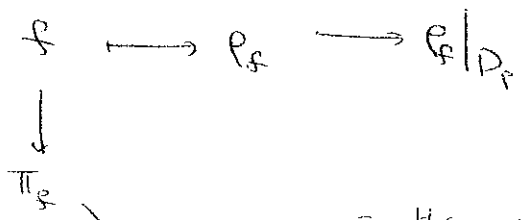
$\rho_{f,p} \cong$ restriction of ρ_f to $\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$

In particular, our former statement about $\rho_f|_{G_{\mathbb{Q}_p}}$ for $p \nmid N$ is a conseq. of this



R_{l₂} : One can almost prove Local Langlands for GL_2 by "writing down both sides"

What about $p=2$?



$Gal(\bar{Q}/Q) \rightarrow GL_2(\bar{Q})$
These are lots of these

\rightarrow He will write down elements!
 $\Pi_{f,p}$ = not that many of these

Q1) Can we recover $\Pi_{f,p}$ from P_f/D_f ?
Yes (T. Saito)

$$\left[\begin{array}{cc} (* & *) \\ (0 & *) \end{array} \right] \left[\begin{array}{cc} (0 & -1) \\ (1 & 0) \end{array} \right]$$

P. Colmez

Q2) Can we go other way? Answer must be "no".

- So what can we do?

- (I) - use global nature of f to put a little extra str. on $\Pi_{f,p}$
- (II) Prove p -adic Local Langlands for GL_2