8

Mar 9. 2006. Thursday. 1:00pm – 2:30pm Kevin Buzzard (8th)

(E.V.)

8-th lecture.

$|U| \leq 1$ r-overconvergent ftns?

(p=2, true)

Recall I talked about

$$W = \tfrac{p-1}{2} \text{ unit disks} \qquad (k, k, \cdots k)$$

$$\cup\!\!\mid$$

W^{+} = half of them $\qquad (k_1, k_2, \cdots, k_g)$

$$\cup\!\!\mid$$

W° = one of them

(p-adic ζ-ftn on W^{+}

(Eisenstein family lives over W°

$\forall \kappa \in W^{\circ}$ there's a power series

exp'n $E_{\kappa} = 1 + \cdots = 1 + \sum_{n \geq 1} \underset{\sim}{a_n} q^n \in \mathbb{C}_p[[q]]$ & $a_n \in \mathcal{O}_{\mathbb{C}_p} \; \forall n \geq 1$.

$E_{\kappa} \in \mathcal{O}_p[[q]]$ & $E_{\kappa} \equiv 1$ in $\overline{\mathbb{F}}_p[[q]]$

($\mathcal{O}_p$ = integers of $\mathbb{C}_p$)

Recall a q-exp'n $F \in \mathbb{C}_p[[q]]$ is an overconvergent modular form of wt $k \in W^0$, if F/E_k is the q-exp'n of an overconvergent modular fctn.

Facts. F wt k, G wt λ $\Rightarrow$ $F \cdot G$ wt $k \cdot \lambda$

If F is overconvergent wt k & T is a Hecke operator

Then TF is overconvergent

Here's one proof

R: any ring.

$U: R[[q]] \longrightarrow R[[q]]$ $\quad U(\sum a_n q^n) = \sum a_{np} q^n$

$V: R[[q]] \longrightarrow R[[q]]$ $\quad V(\sum a_n q^n) = \sum a_n q^{np}$ $\quad U \circ V = id$ $(V \circ U \neq id)$

& more generally

$U(F \times V(G)) = G \times U(F)$. $\quad F, G \in R[[q]]$.

Recall there's Hecke operator U on overconvergent modular fctns.

Let me explain why, if F is overconvergent $\left(\dfrac{E_k}{(E_1)^k}\right.$ overconvergent$\left.\right)$ wt k, then $U(F)$ is too.

It's a theorem of Coleman that if V_k denote $V(E_k)$

then E_k/V_k is the q-expansion of an overconvergent modular ftn.

[Q) How far does it overconverge?]

Using this thm, we easily deduce that U preserves wt k forms

Say F is overconvergent wt k

Then $F = E_k \cdot G$, G overconv wt 0.

$$\therefore U(F) = U(G \cdot E_k)$$

$$= U\left(G \cdot \frac{E_k}{V_k} \cdot V_k\right) = E_k \times \underbrace{U\left(G \cdot \frac{E_k}{V_k}\right)}$$

overconv. mod ftn

$\therefore$ $\dfrac{U(F)}{E_k}$ is overconvergent wt 0. $\quad \therefore U(F)$ is overconvergent wt k

This argument proves that U is cpt on r-overconvergent form of wt k as long as r is suff. small

(r p^{-r} 0-overconvergent means ftn on ordinary locus) because if G is ε-overconvergent, then $G \times \frac{E_\kappa}{V_\kappa}$ is too.

$\therefore$ U of it is $p\varepsilon$-overconvergent.

Now restart back to ε-overconvergent

Argument shows this

If ε is small define ε-overconv. wt κ forms

$:=$ q-exp'ns (F_κ) s.t. F/E_κ is ε-overconvergent.

U is a cts map on

$$\begin{tikzcd} \varepsilon\text{-overconvergent map} \arrow[r, "U"] \arrow[dr, "\text{cpt}"'] & p\varepsilon\text{-overconvergent forms} \arrow[d, "\text{res (cpt)}"] \\ & \varepsilon\text{-overconvergent forms} \end{tikzcd}$$

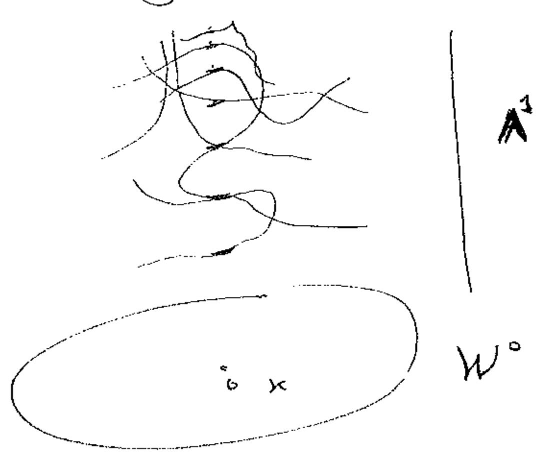
$\therefore$ U-acting on ε-overconvergent forms wt κ has a char. power series $P_\kappa(T) = \det(1 - T\cdot U)$

Just as in wt 0, this cps is indep of $\varepsilon > 0$.

Here's an idea then:

For any $\kappa \in W^\circ$, plot the zeroes of $P_\kappa(T)$ in $\mathbb{A}^1/\mathbb{C}_p$ (rigid space)

As κ varies, you get a subset of $W^\circ \times \mathbb{A}^1$



Rk. $E_0 = 1$ & if κ is close to 0,

$E_\kappa \equiv 1$ mod big power of p

This picture is called the spectral curve associated to p

How does one can construct it properly?

Remember the Lecture when I did Serre's Endomorphisms' paper?

I defined a CPS for a cpt operator on an ON-able p-adic Banach space

We want to generalize this to ON-able Banach modules over a complete ring of some kind.

eg. Let R be the ring $\mathbb{Q}_p\langle T\rangle$. Norm on R. $|\sum a_n T^n| = \max|a_n|$

Can define a Banach module over R as a complete normed R-module M + axioms $|rm| \le |r||m|$ etc

Key example: an ON able one.

Pick a set I, eg $I = \{1, 2, 3, \dots\}$

Set M = ftns $f: I \to R$ s.t $f(i) \to 0$ as $i \to \infty$

& $|f| = \max_i |f(i)|$

One defines cts & cpt operators on such things

Finite rank: $\operatorname{Im}\phi \le$ fin. -gen. R-module

Cpt: limit of finite rk

Cpt ops have a CPS ~~at~~ some df. $\in R[[T]]$

<u>Idea</u>: if $D \subset \mathcal{W}^0$ is a small closed disk, let's define the $\mathcal{O}(D)$-module of ε-overconvergent forms of wt D to be the ftns on $D \times X_1(N)_{\geq p^{-\varepsilon}}$
$\cup \atop \infty$

[Remark. I am thinking about an overconvergent wt k form as being <u>equal</u> to an overconvergent ftn on $X_1(N)^{ord}$]

Define the q-expansion of such an object as being the obvious q-exp'n on $\mathcal{O}(D)[[q]]$

$\times E_D$ where E_D = q-expansion of Eisenstein family over D

$$E_k = 1 + \sum_{n \geq 1} a_{n,k} q^n$$

$a_{n,k} = A_n(k)$ where A_n is a fctn on $\mathcal{W}^\circ$

$$\left(A_1 = \frac{2}{\zeta_p} \quad . \quad A_n = \frac{2}{\zeta_1} \sigma_n^* \right)$$

$$\sum_{\substack{0<d \mid n \\ p \nmid d}} \frac{k(d)}{d}$$

$\therefore$ can think of all Eisenstein family at once as being an element of $\mathcal{O}(\mathcal{W}^\circ)[[q]]$

$$1 + \sum_{n \geq 1} A_n q^n$$

The fctn $k \mapsto k(d)$ is a fctn on wt space

$k(1+p)$

$k(d) = k((1+p)^t)$

$(1+w)^t$

In fact one can check that W is the usual parameter on $\mathcal{W}^\circ$ space.

$$\kappa(\underset{1+4 \text{ if } p=2}{1+p}) = w+1.$$

then $\mathbb{E} \in \mathbb{Z}_p[[w]][[q]]$

This is a "<u>computable object</u>"

In fact $\frac{2}{\zeta_p} \in w\mathbb{Z}_p[[w]]$ (pole of ζ at $w=0$)

$$\therefore \mathbb{E} \in 1 + wq\mathbb{Z}_p[[w, q]].$$

<u>Define</u> $\mathbb{V} = V(\mathbb{E})$

$$= \mathbb{E}(q^p) \in 1 + wq\mathbb{Z}_p[[w, q]]$$

$$\mathbb{E}/\mathbb{V} \in 1 + wq\mathbb{Z}_p[[w, q]]$$

& one can now specialize to $w = w_0 \in W$. $w_0 \leftrightarrow k$

& recover E_k / V_k.

Explicit analysis of <u>2-adic</u> spectral curve near boundary of <u>W°</u>.

Exciting new parameter!

E_2 = classical <u>wt 2</u> <u>level 2</u> Eisenstein Series

$$1 + 24(q + q^2 + \cdots)$$

$V_2 = V(\overset{1+24q^2+\cdots}{\overset{\parallel}{E_2}})$ classical wt 2 level 4

$$\frac{E_2}{V_2} = 1+24+\cdots \quad \text{meromorphic fn on } X_0(4)$$

Set $$y = \frac{\frac{E_2}{V_2}-1}{24} = q - 20q^3 + 462q^5 + \cdots \in \mathbb{Z}[[q]]$$

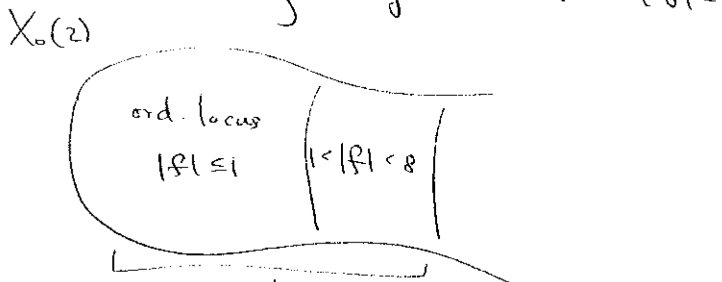
$$y : X_0(4) \xrightarrow{\sim} \mathbb{P}^1$$

Recall $f = \frac{\Delta(q^2)}{\Delta(q)} : X_0(2) \longrightarrow \mathbb{P}^1$ & one checks that

$$f = \frac{y+8y^2}{(1-8y)^2}$$

In particular, the natural map $X_0(4) \to X_0(2)$ induces an isomorphism between region $|y|\le 1$ & $|f|\le 1$.

& more generally between $|y| \le \alpha$ & $|f| \le \alpha$ for $\alpha < 8$



disc: f & y are parameters

Now we use powers of y instead of f

What is the matrix of U on overconvergent wt κ forms w.r.t basis $V_\kappa, V_\kappa(cy), V_\kappa(cy)^2, \dots$ $\quad c \in \mathbb{C}_2$

One can answer this question $\Longleftrightarrow$ one knows how to write E_κ/V_κ as a power series in y

Here's what I know.

If $\kappa \in \mathcal{W}^\circ$ & the parameter $w = w(\kappa) = \kappa(5)-1$ satisfies $|w| < \frac{1}{8}$.

then Kilford & I showed that E_κ/V_κ was in $\mathcal{O}_2[[8y]] = \mathcal{O}_2[[8f]]$

Next time I'll show you why.

Next time I'll show you why

In fact we really prove that

$$\mathbb{E}/\mathbb{V} \in \mathbb{Z}_2[[w,y]] \text{ is also in } \mathbb{Z}_2[[\tfrac{w}{8}, 8y]]$$

$$|w|<1,\ |y|<1 \qquad\qquad |w|<\tfrac{1}{8}\quad |y|<8$$

- $\mathbb{E}/\mathbb{V} \in \mathbb{Z}_2[[w,y]] \cap \mathbb{Z}_2[[\tfrac{w}{8}, 8y]]$

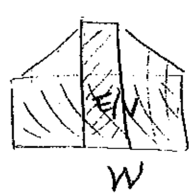
$$= \mathbb{Z}_2[[w, wy, 8y]]$$

<u>Consequence</u>: If $w = w_0$, $w_0 \in \mathbb{C}_2$, $1 > |w_0| > \frac{1}{8}$, $w_0 \leftrightarrow \kappa$

then $E_\kappa/V_\kappa \in \mathcal{O}_2[[w_0 y]]$

What happened is that E_κ/V_κ <u>being</u> <u>very</u> overconvergent in center of wt space

$\Rightarrow$ a little overconvergent at boundary.



Moreover, if $E_\kappa/V_\kappa = g_\kappa(w_0 y)$, $g_\kappa \in \mathcal{O}_2[[X]]$

then for $|w_0| > \frac{1}{8}$, $\overline{g_\kappa} \in \mathbb{F}_2[[q]]$ is independent of κ!

as one checks easily that $\overline{g_\kappa} = \sum_{n \geq 0} \overline{a_{nn}} X^n$

$$\text{if } \quad \mathbb{E}/\mathbb{V} = \sum \underset{\mathbb{Z}_2}{\underset{\cap}{a_{ij}}}\, w^i y^j$$

One can even compute $\overline{g_\kappa}$ by choosing <u>one</u> $\kappa \in W$

near boundary & bashing it out

e.g. choose $\kappa: \mathbb{Z}_2^\times \longrightarrow \mathbb{C}_2^\times$

$$\kappa(x) = \begin{cases} x & x \equiv 1\ (4) \\ -x & x \equiv 3\ (4) \end{cases}$$

$\kappa \longleftrightarrow$ classical point (κ, χ), $k=1$, χ conductor 4

$$w = \kappa(5) - 1 = 4, \quad |w| = \tfrac{1}{4} > \tfrac{1}{8}$$

One checks that $E_\kappa = \sum_{a,b \in \mathbb{Z}} q^{a^2+b^2} = 1 + 4q + \cdots$

$V_k = E_k(q^2)$ level 8

$h = E_k / V_k$ is a modular fn of level 8

— want this as a power series in y

One checks that $4yh^2 + (1+8y)h + (1+8y) = 0$

Can solve!

$$\bar{g}_k = \sum_{n \geq 0} x^{2^n - 1}$$

Just as in wt 0, one gets ones teeth & compute $U((cy)^n)$ as a power series in cy

Then $U(V_k (cy)^n) = E_k U((cy)^n) = \frac{E_k}{V_k} V_k U((cy)^n)$

Now get a "formula" for matrix entries "$g_k(cy)$"

— for some entries all we know is a lower bdd on val'n

For some entries we know what valuation is.

Big matrix representing U in wt k looks like

& one checks easily that one can throw away all 0 rows & corresponding columns & not change CPS.

The new matrix has the property that

w^i divides i^{th} row & furthermore, if you divide i-th row by w^i,

the resulting matrix has the property that det of top left hand $n \times n$ chunk is always a unit.

$\Rightarrow$ Slopes of CPS are $1, v(w), 2v(w), 3v(w), \ldots$

Pictures of 2-adic spectral curve $\longrightarrow$