Mar 16. 2006. Thursday. Kevin Buzzard (E.V.) 10th lecture

10th lecture (1:00–2:30 PM)

Recall. last time I said

for κ near boundary of 2-adic wt sp.

$\kappa \leftrightarrow w \in \mathcal{W}^0$, $|w| > \frac{1}{8}$.

the norms of evals of U_2 on wt κ forms were

$1, |w|, |w|^2, \cdots$

If κ is "classical", $\kappa \mapsto (x \mapsto x^k \chi(x))$

then classical forms wt k, character χ $\hookrightarrow$ overconvergent forms with wt κ

Cpt. general p.

WLOG $k \geq 1$, then E_k is a classical MF wt k char χ & it's q-expansion $\equiv 1$ mod max. ideal of integers of $\mathbb{Q}_p(\chi)$

This implies that E_k never vanishes on $X_1(p^n)^{ord}$ & hence on $X_1(p^n)[r]$ for some $r > 0$

Hence if f is classical wt k, char χ

f/E_k is mero wt 0 level p^n

& holo. on $X_1(p^n)[r]$ $\therefore$ holo. on $X_0(p^n)[r] = X_0(1)[r]$

Standard thm:

if $f \in M_k(\Gamma_1(N))$, $p \mid N$.

then either U_p-eigenvalue of f is zero

or it's a non-zero alg. integer α, $\pm p^{\frac{k-1}{2}}$

& $\exists$ alg integer β s.t. $\alpha\beta = p^{k-1}$ $\quad X^2 - a_p X + p^{k-1}$

Hence $0 \le v_p(\alpha) \le k-1$.

Hence classical U-e.vals all show up @ beginning of list

$p=2$, $N=1$.

By a counting argument the classical e.values are an initial segment of the list & overconvergent U-evalue is classical $\Leftrightarrow$ val'n $\le k-1$ $\quad$ ($\mathrm{cond}(\chi) \ge 4$ $k \in \mathbb{Z}$ $(-1)^k = \chi(-1)$)

In fact, Coleman proved that if f is overconvergent wt $\kappa = (k, \chi)$ & $Uf = \alpha f$ $\quad v(\alpha) < k-1$

then f is classical. $\quad$ (Caution: not $\le$)

In fact, there do exist overconvergent, nonclassical f with U-e.value val'n $= k-1$
& $\exists$ classical f, too.

Another funny thing:

Coleman constructs a map from overconvergent of wt-k to ~~overconvergent~~ forms wt $k+2$
called Θ^{k+1} $\quad$ $k \ge -1$ $\quad$ $(0, -k-1)$

On q-expansion. $\quad$ $(k+1, 0)$

$$\Theta^{k+1}\left(\sum_{n\ge 0} a_n q^n\right) = \sum_{n \ge 0} n^{k+1} a_n q^n$$

Note that if F is an eigen form wt $-k$, then $\Theta^{k+1} F$ is an eigen form of wt $k+2$
& $V(\Theta^{k+1} F) = p^{k+1} a_p$

$|a_p| \le 1$. $\therefore$ val'n of U-e.val is $\ge$ wt -1

Payman Kassaei later gave a beautiful simple re proof of Coleman's result.
generalizes to Shimura curve

Finally there's a definition of an overconvergent automorphic form of wt $\kappa \in W$.

Recall
$$W = W^{0} \times \mathrm{Hom}(G, \mathbb{C}_p^{\times})$$
G = finite gp $= \begin{cases} (\mathbb{Z}/4\mathbb{Z})^{\times} & \text{if } p=2 \\ (\mathbb{Z}/p\mathbb{Z})^{\times} & \text{if } p>2 \end{cases}$

$p \nmid N$

I've defined $X_1(N)[r]$

$0 \le r < \frac{p}{p+1}$.

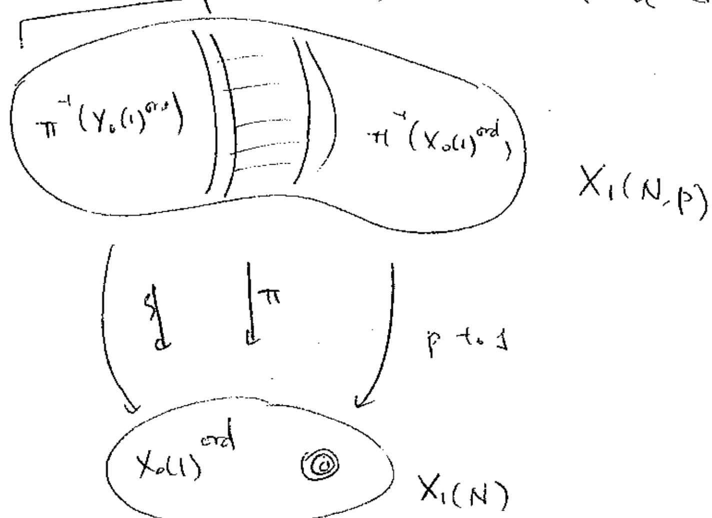
An elliptic curve corresponding to a point in $X_1(N)[r]$ (r in above range) has a canonical subgp of order p.

$\therefore$ Get a section
$$X_1(N)[r] \longrightarrow \overset{\Gamma_1(N)\cap\Gamma_0(p)}{X_1(N,p)}$$
of forgetful map π

Define $X_1(N,p)[r]$ = image of $X_1(N)[r]$

Note that $\pi^{-1}(X_1(N)[r])$ has 2 cpts, namely $X_1(N,p)[r]$ & another mapping down to $X_1(N)[r]$ via a degree p map

$X_0(2)[r]$



Let $\psi : X_1(Np) \to X_1(N,p)$ be the natural map

$$(E, \underset{N}{P}, \underset{p}{Q}) \to (E, P, \langle Q\rangle)$$

Define $X_1(Np)[r] = \psi^{-1}(X_1(N,p)[r])$

More generally

~~For~~ any modular curve X admitting a natural ~~forgetful~~ map ψ to $X_1(Np)$

(e.g. $X_1(Np^n)$ or $X_1(N,p^n)$)

Define $X[r] = \underbrace{\psi^{-1}(X_1(N,p)[r])}_{\text{conn. cpt of containg } \infty}$

__Rmk__: if $0 \le r < \frac{p}{(p+1)p^{t-1}}$

then $X_1(N,p^t)[r] = X_1(N,p)[r] = X_1(N)[r]$

because if $r < \frac{p}{(p+1)p^{t-1}}$ then an ell. curve in $X_0(1)[r]$ has a canonical subgp of order p^t.

Set $q = \begin{cases} p & p>2 \\ 4 & p=2 \end{cases}$

Now $X_1(Nq)[r]$ has an action of $(\mathbb{Z}/q)^*$ & the quotient is $X_1(N,q)[r] = X_1(N)[r]$ if r is suff. small

__Defn__. If $\kappa \in \mathcal{W}$ then an $\overset{r-}{\text{overconvergent}}$ MF wt κ & level $\Gamma_1(N)$ $(p \nmid N)$ is a formal q-expansion $F \in \mathbb{C}_p[[q]]$

s.t. if $\kappa = (\kappa_0, \chi) \in \mathcal{W}^0 \times \mathrm{Hom}((\mathbb{Z}/q)^*, \mathbb{C}_p^*)$

then F/E_{κ_0} is the q-exp. of a ftn on $X_1(Nq)[r]$ in the χ-eigenspace for the Diamond operators.

Spectral Curve & eigen Curve

We have an Eisenstein family

$$\mathbb{E} = 1 + \cdots \quad \in \mathcal{O}(W^\circ)[[q]]$$
$$\cup\!\!\!\mid$$
$$\mathbb{Z}_p[[w]]$$

$$\mathbb{E} = 1 + \frac{2}{\zeta^*} q + \cdots$$

Classical result of p-adic L-ftn

In fact $\mathbb{E} \in \mathbb{Z}_p[[w]][[q]]$

Now let $D \subset W^\circ$ be a closed disk.

If $k \in D$, then E_k (specialization of $\mathbb{E}$ at k) divided by $V(E_k)$ is an overconvergent ftn.

As D is closed one can concoct an r s.t $E_k/V(E_k)$ is r-over convergent & has no zeros for all $k \in D$, $r = r(D)$

Def. If $0 \le s \le r(D)$

an s-overconvergent modular form of wt D is a formal power series

$F \in \mathcal{O}(D)[[q]]$ s.t. if $\mathbb{E}_D$ denotes the restriction of $\mathbb{E}$ to $\mathcal{O}(D)[[q]]$ $1 + \cdots$

then $F/\mathbb{E}_D$ is the q-expansion of a fn on $X_1(N)[s] \times D$

$$\cup\!\!\!\mid$$

Function on here has an expansion $\quad$ open disk $0 \le |q| < 1$ $\times D$

in $\mathcal{O}(D)[[q]]$

More generally, if D is any rigid space over $\mathcal{W}$ & the image of D in $\mathcal{W}^\circ$ via projection $\mathcal{W} \to \mathcal{W}^\circ$ has the property that $\exists r$ as above

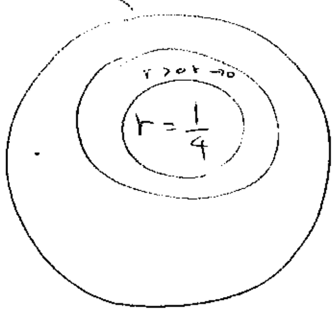
s.t. E_d/V_d is r-overconvergent $\forall d \in D$

then $\exists$ def'n of an s-overconvergent MF of wt D

$$X_1(Np)[s]$$

Even more generally, if

$D \to \mathcal{W}$ is any rigid space over $\mathcal{W}$ one can define an overconvergent **MF** of wt D by writing down an admissible cover of D by affinoids & demanding that you are r_X-overconvergent $\forall_{\text{affinoid } X}$.



Hecke ops act on overconvergent forms of wt D & also on r-overconvergent forms wt D if E_k/V_k is r-overconvergent $\forall k \in \mathrm{Im}(D)$.

Let D be an affinoid,

$D \to \mathcal{W}$.

Let r be $\leq r(D)$

This is an ONable Banach space over $\mathbb{C}_p$.

$$\left\{ \begin{matrix} r\text{-overconvergent forms} \\ \text{of wt } D \end{matrix} \right\} \cong \mathcal{O}(X_1(N)[r] \times D)$$
$$= \underline{\mathcal{O}(X_1(N)[r])} \,\hat{\otimes}\, \mathcal{O}(D)$$

Hence

$\mathcal{O}(X_1(N)[x]) \hat{\otimes} \mathcal{O}(D)$ is an ONable Banach module over $\mathcal{O}(D)$

a complete normed ring

$\exists$ theory of dts & cpct ops for Banach modules in this generality

& U is compact.

& one gets a CPS.

$CPS(U) \in \mathcal{O}(D)[[T]]$. which converges for all T.

independent of $r > 0$.

$CPS(U) = \sum_{i \geq 0} f_i T^i \quad f_0 = 1 \qquad |f_i| \to 0$ v. quickly

$|f_i|\lambda^i \to 0 \quad \forall \lambda \in \mathbb{R}_{\geq 0}$.

If $D \subseteq W^\circ$. & $k \in D$

then the specialization of $CPS(U$ on OCMFs $_{\text{wt } D})$

$= CPS(U$ on OCMFs $_{\text{wt } k})$

More generally

if
$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[d] \\ & W \end{tikzcd}$$
are rigid spaces

then the pull back of $CPS(U$ on OCMFs wt $y)$

$= CPS(U$ on OCMFs wt $x)$

$k \in \mathbb{Z}$, $k \geq 2$

$k \notin W^\circ$ in general

(W^-) forms, if $N > 1$

odd

F classical wt k level Np

char $\chi = \chi_N \chi_p$

then F is overconvergent

wt $x \mapsto x^k \cdot \chi_p(x)$

$\chi_p(-1) \overset{?}{=} (-1)^k$

$\mathrm{Hom}\left(\left(\varprojlim \left(\mathbb{Z}/Np^r\mathbb{Z}\right)^\times\right) \to \mathbb{C}_p^\times\right)$

Spectral curve is the graph of $CPS(U)$ on overcgt MFs wt W.

i.e. zeros of $CPS(U$ on OCMFs wt $W)$
$\in \mathcal{O}(W)[[T]]$

Better in $\mathcal{O}(W \times \mathbb{A}^1)$

$\therefore$ its zero set is a closed subset of $W \times A'$

(w, α)

(w,α) is in $\Leftrightarrow$ α is a zero of $CPS(U)$

$\mathbb{Q}$

wt w OCH fns

$1 + \cdots \in \mathbb{Q}_p[[T]]$

$\Leftrightarrow$ $\frac{1}{\alpha}$ is an e. val. for U.

Zero set $\subseteq W \times \mathbb{G}_m$

& (w, α) in $\Leftrightarrow$ $|\alpha| \geq 1$

Eigen Curve

General Setup

* A a reduced affinoid $\quad D = \mathrm{Max}(A)$

  e.g. $\mathcal{O}(D)$. $D \subseteq W$ closed disc.

* M an ONable Banach module over A
* Collection of commuting elts of $\mathrm{End}_A^{\mathrm{cts}}(M)$

  $U, t_1, t_2, t_3, \cdots$

  U : cpt.

Goal: build a cover of the spectral variety

$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad ||$

zero set of $CPS(U)$.

$\cap$

$\mathcal{O}(D \times A')$

that will see e.vals of all the t_i too.

Say we factor $CPS(U)$

$= P(T) = Q(T) \times S(T)$

where $Q(T)$: a polynomial of deg n with leading term a unit in A.

$(Q(T), S(T)) = \mathcal{O}(D \times \mathbb{A}^1)$

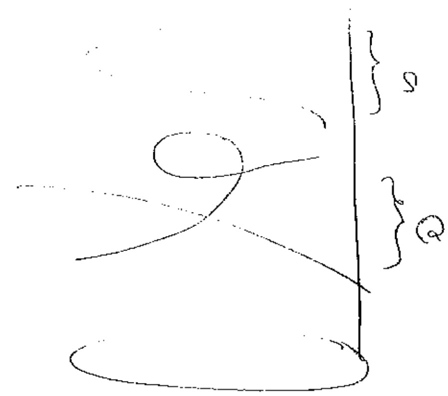
i.e zeros of $P(T)$ is disjoint union of zeros of $Q(T)$ & zeros of $S(T)$

Then $M = N \oplus F$, $N = eM$, $e \in A[U]$

U-invariant decomposition

N is projective of rk r

(CPS(U) on N is Q
CPS(U) on F is S



All t_i commute with U

$\Rightarrow$ N & F are t_i-invariant too

$\therefore$ all the t_i act on N

Let $\mathbb{T}$ = sub-A-alg of $\mathrm{End}_A N$ gen. by the t_i and U

$\mathbb{T}$ is finite over A

$\therefore$ $\mathbb{T}$ is also an affinoid.

& $\mathrm{Max}(\mathbb{T})$ is a rigid space mapping down to D

"Glue the $\mathrm{Max}(\mathbb{T})$ together as D varies
& as ~~ff~~ factorization varies.

$\mathbb{T}$ is naturally an A-algebra.

but in fact $\underset{U}{\overset{\mathbb{T}}{\cup}}$ is naturally an $A[T]/Q(T)$-algebra

via the map $T \mapsto U^{-1}$ (on N) $\mathrm{Max}(A[T]/Q(T))$ is a chunk of the spectral variety.

So
$$\begin{array}{ccc} \mathrm{Max}(\mathbb{T}) & \longrightarrow & \text{Spectral variety} \\ & \searrow \quad \swarrow & \\ & D & \end{array}$$