Mar 21. 2006 Tuesday Kevin Buzzard. (EV)
Review of Eigencurve 11th lecture

①. the notion of an overconvergent mod form of wt κ or wt D. DCW
+ fact that these spaces have an action of a commuting set of Hecke oprs including U cpt

② A purely formal machine which gives an ON-able Banach **module over** an affinoid
+ commuting linear maps. one of which is (U) cpt.
gives out a parameter space for the systems of eigen values + condition that $\forall$ U eigen value is nonzero

② in context + example

$\boxed{\text{Idea}}$: If S is a classical space of cusp forms & $\mathbb{T}_{\mathbb{C}}$ is associated Hecke alg
then there's a natural pairing

$$S \times \mathbb{T} \to \mathbb{C}$$
$$(f, T) \mapsto a_1(f|T)$$

Identifies S with

$$\mathrm{Hom}_{\mathbb{C}\text{-module}}(\mathbb{T}, \mathbb{C})$$
$$\cup\!\mid$$
$$\mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{T}, \mathbb{C}) = \text{ring hom}$$

Associated subset of is the normalized eigenforms
Set of normalized eigenforms on S is $\mathrm{Spec}(\mathbb{T}_{/\mathbb{C}})$

Def. A Tate algebra is a ring of the form
$$K\langle T_1, T_2, \ldots T_n\rangle$$
K field complete w.r.t. non-trivial non-arch norm

& $K\langle T_1 \cdots T_n \rangle = \sum a_i T^i$ s.t. $a_i \to 0$ as $i \to \infty$

Noetherian ring UFD

<u>Norm</u>. $|\sum a_i T^i| = \max |a_i|$

Any ideal of $K\langle T_1 \cdots T_n\rangle$ is closed & maximal ideals have finite K-codimension.

If $K = \bar{K}$, then $\mathrm{Max}(K\langle T_1 \cdots T_n\rangle) = \{(z_1 \cdots z_n) \in K^n,\ |z_i| \le 1\ \forall i\}$

$(z_i) \mapsto (T_i - z_i)$

An affinoid algebra B a ~~quaternion alg~~ quotient of Tate alg by a (closed) ideal

If $A = K\langle T_1 \cdots T_n\rangle/(f_1 \cdots f_m)$ & $K = \bar{K}$ then $\mathrm{Max}(A) = \{\underline{z} \in B^n : f_i(z) = 0\ \forall i\}$

(BGR Schneider in LMS Durham volume)

<u>Example</u> K, $K\langle T\rangle$ PID | A: an affinoid alg

$K\langle 1, S\rangle \supseteq (S)$, | $\mathrm{Max}(A)$ = affinoid space = analogue of affine variety

General rigid space: glue affinoids

Idea from last time:

M: an ONable Banach module over an affinoid alg A

$U: M \to M$ a cpt A-linear map

$\mathrm{CpS}(U) = \det(1 - tU) = 1 + c_1 t + c_2 t^2 + \cdots$

If $\mathrm{CpS}(U) = Q(t)\, S(t)$

$Q(t) = 1 + \cdots + a_n t^n$, $a_n \in A^\times$

& Q, S coprime in $A\{\{t\}\}$

(i.e. zero sets of Q & S in $\mathrm{Max}(A) \times \mathbb{A}^1$ are disjoint) then $M = N \oplus V$ a U-invariant decomp. N: projective of rk n

$\mathrm{Ops}(U \curvearrowright N)$ is $Q(t)$ $\qquad N = eM$

If we also have commuting ops $T_1 T_2 \cdots$ $\qquad e \in \overline{A[U]} \subseteq \mathrm{End}_A(M)$

then $\mathbb{T}$ = A-sub alg. $\mathrm{End}_A(N)$ generated by U & all T_i

then $\mathbb{T}$ is finite over A

(not nec. flat)

$\mathrm{Max}(\mathbb{T})$ is parametrising systems of

e. vals in N . "make it bigger"

$\Rightarrow$ paramet'ing system of e.vals in M with U-e val. a unit.

$m \in A$

$\otimes A/m$ $\quad M \otimes A/m$ is Ban. space over a field

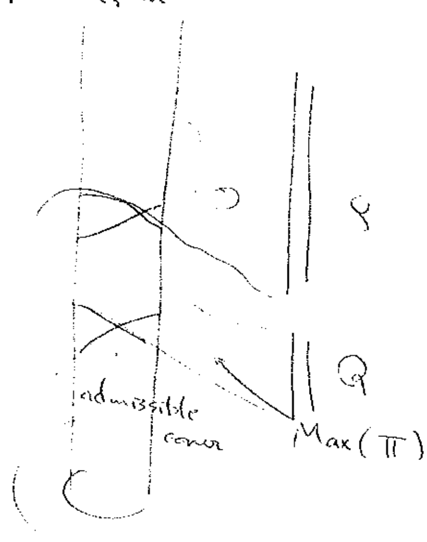
$m \in M \otimes A/m$ a non-zero eigen vector for all T_i & U. $U_m \neq 0$

Then $\forall i$. $\exists c_i \in K$. $T_i m = c_i m$

$(T_i - c_i)_i$ = maximal ideal in $\mathbb{T}$

$$\mathrm{Max}(\mathbb{T}) \searrow \mathrm{Max}(A[t]/Q(t)) \to \mathrm{Max}(A)$$

$U^{-1} \mapsto t$



Examples of formal construction

Say $A = k\langle X \rangle$. $M = A^2$

$U = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ $\quad a, b \in A$ $\quad \det(1 - tU) = (1 - ta)(1 - tb)$

Zero locus of $\det(1 - tU)$ = union of graph ~~of~~ $\begin{cases} at = 1 \\ bt = 1 \end{cases}$

in $\mathrm{Max}(A) \times t$

e.g. $a = X + 1$

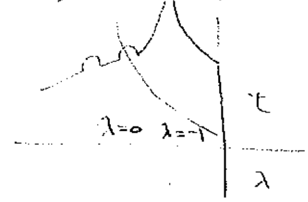
$b = X$

Say $\lambda \in K$, $|\lambda| \le 1$, & let $m = X - \lambda$

Let's specialize everything to A/m

by $\otimes$ with A/m. $M/m = K^2$; U on M/m is $\begin{pmatrix} 1+\lambda & 0 \\ 0 & \lambda \end{pmatrix}$

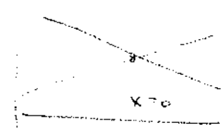
$CPS(U)$ is $(1 - t(1+\lambda))(1 - t\lambda)$



$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are 2 families of eigenvectors.

2) $U = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ $\quad a = 1 + X$, $b = 1 - X$ $\qquad a = p^{s} + X$, $b = p^{s} - X$

Spectral variety = graph of $CPS(U) = 0$



At $X = 0$, situation specializes to $U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad U - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} p^{s}$

Say T is another Hecke operator

$T = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$ & $c(0) \ne d(0)$

$CPS(U) = Q(T)\, S(T)$

$ \overset{\shortparallel}{1}$

$\mathrm{End}(M) \supseteq$ sub-alg gen. by U & $T = \mathbb{T}$

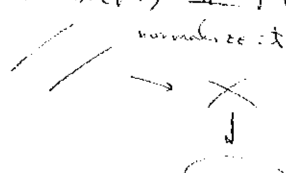
$= A \oplus A \ne A[U]$

classical at $v = k-1$, overconvergent

U: nonsemi-simple

& $\mathrm{Max}(\mathbb{T}) = \mathrm{Max}(A) \amalg \mathrm{Max}(A)$

normalize it



f: classical wt 1

$p \nmid$ level

$\rho_f(\mathrm{Frob}_p)$ scalar

$X^2 - a_p X + \chi(p) = (X - \alpha)^2$

$\left[\begin{array}{ll} e_1 & f(q) - \alpha f(q^p) \\ e_2 & f(q^p) \end{array}\right.$

$U e_1 = \alpha e_1$

$U e_2 = f(q) = e_1 + \alpha e_2$

$\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$

Last example

$A = k\langle X, Y\rangle$, $U = \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}$, $T = \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix}$

$CPS(U) = (1-t)^2$: Spectral variety is $\mathrm{Max}\left(A[t]/(t-1)^2\right)$

Eigenvariety $= \mathrm{Max}(A \oplus I\cdot\varepsilon)$ blow up

not flat

$\downarrow$

$\mathrm{Max}(A)$ not reduced $\varepsilon^2 = 0$ $I = (X, Y)$

Facts about eigen curve = what you get by gluing all the $\mathrm{Max}(\mathbb{T})$s in case where A = "wt space & M = over conv M.F.

(1) Classical pt are everywhere

Because components of eigencurve map down to wt sp surjectively except perhaps missing a finite set

$\Rightarrow$ hits lots of classical weights

& If f is a point of wt $\kappa = (k, \chi)$

then for $\kappa' = (k', \chi)$ $k' \equiv k \ ((p-1)p^N)$ & $k' \gg 0$

then near f will be overconvergent small slope

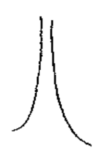
f' $\Rightarrow$ f' classical

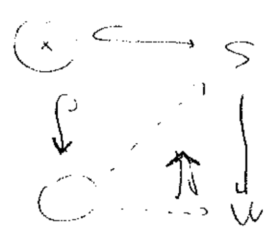
Eigen curve "interpolating" finite slope classical eigen forms

$U f \neq 0$

Are there holes ?

If $p=2$ & $N=1$





no holes (explicit computation)

- spectral curve has no holes

Eigen curve is locally a finite cover

$\therefore$ no holes either

valuative criterion

Associated to a classical point is a Galois rep'n

These "interpolates" & in fact to any overcgt finite slope eigen form, there's a Galois rep'n

N = tame level

$p \nmid N$

$\mathcal{C}$ eigen curve

$= \bigcup \mathrm{Max}(\mathbb{T})$

If $n \geq 1$, then the T_n is a map

$\mathcal{C} \dashrightarrow \mathbb{A}^1$

$\subset W$

& if $f \in \mathcal{C}(\mathbb{C}_p)$

then $\exists \rho_f : G_{\mathbb{Q}} \to GL_2(\mathbb{C}_p)$

unram. outside Np

$\rho_f(\mathrm{Frob}_\ell)$ has trace $T_\ell(f) \in \mathbb{A}^1(\mathbb{C}_p) = \mathbb{C}_p$

& det $\chi_p \times (x \mapsto \frac{\kappa(x)}{x})$

tame character

<u>Rk</u>. An overconv. finite eigenform is "global" (as $\exists$ Gal rep) but not in general <u>Motivic</u> ($\nexists$ ℓ-adic Gal rep'n for any $\ell \neq p$)

ρ_f has one H-T wt 0 other k-1

Kisin: $D_{cris}(\rho_f)$ has dim ≥ 1

One can write $\mathcal{C} = \mathcal{C}^{cusp} \cup \mathcal{C}^{Eis}$

(union is disjoint if $N=1$ & p is regular)

$\mathcal{C}^{cusp}$ = what you get if you apply machine to cuspidal modular forms (vanish @ all cusps on $X_1(N)^{ord}$)

$\mathcal{C}^{cusp}$ represents a functor over W

sending a rigid space $X \to W$

to the set of normalized ($F = q + \cdots$) cuspididal overcgt eigenforms of wt X with U-eigenvalue in $\mathcal{O}(X)^\times$

<u>Waffle</u> $\mathcal{C} \supset$ dense set of classical new forms

A classical eigenform gives rise to an automorphic repn of $GL_2(\mathbb{A})$

Let f be a classical eigen form $f: \mathfrak{H} \to \mathbb{C}$

Want to construct a fcn $\varphi: GL_2(\mathbb{A}) \to \mathbb{C}$ s.t. ~~

Here's how to do it

$$\overset{g}{GL_2(\mathbb{A})} = \overset{\gamma}{GL_2(\mathbb{Q})}\ \overset{k}{K} \cdot \overset{u}{GL_2^+(\mathbb{R})}$$

$\subseteq GL_2(\hat{\mathbb{Z}})$ analogue of $\Gamma_1(N)$ $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \bmod N$

Define $\varphi(g)$ $\quad j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau\right) = c\tau + d$

$$= \sqrt{\det(u)}^{s}\ j(u, i)^{-k} f(u i)$$

$\in \mathbb{C}$ anything

There's a choice

$GL_2^+(\mathbb{R}) \subset \mathfrak{H}$

Any $s \in \mathbb{Z}$ gives you an "**algebraic**" automorphic form

$\Downarrow$

$\exists$ Gal. repn

auto rep $\pi_{f,s}$

$\rho_{\pi_{f,s}} = \rho_f \otimes \chi^s$ $\quad \chi$: cyclotomic character

HT wts are $(s, k-1+s)$

$\left\{ \begin{matrix} \text{All auto} \\ \text{repn for} \\ GL_2(\mathbb{A}) \end{matrix} \right\} \supseteq \left\{ \begin{matrix} \text{cuspidal} \\ \text{algebraic} \\ \text{ones} \end{matrix} \right\} = \left(\begin{matrix} \text{alg} \\ \text{Maass forms} \end{matrix} \right) \amalg \left(\text{new forms} \right)$

big set

f cuspidal Maass form

ρ_f $\pi_{f,s}$

eigen, even

Gal

$\rho_f: G_{\mathbb{Q}} \to GL_2(\mathbb{C})$

(+ bunch of pts)

$\left(\mathcal{C}^{\text{cusp}} \times \mathcal{W}\right)$

$(f \times s)$

the object is interpolating certain automorphic forms for GL_2

$\mathcal{W} \times \mathcal{W}$

$= \operatorname{Hom}(\text{torus in } GL_2(\mathbb{Z}_p), \mathbb{C}_p^*)$

G: reductive gp $\quad$ Is this a right picture?

Automorphic forms for G $\supseteq$ alg. Auto forms $\subseteq$ p-adic guys ?