April 20. 2006 1:00pm Kevin Buzzard (E.V.) 135
(Thursday)
17 th lecture

f: cuspidal eigenform $\rightsquigarrow$ $\rho_f : G_{\mathbb{Q}} \xrightarrow{\text{irreducible}} GL_2(\overline{\mathbb{Q}}_p)$

$$\begin{array}{ccc}
f & & \rho_f \\
\downarrow & & \downarrow \\
\pi_{f,p} \circlearrowleft GL_2(\mathbb{Q}_p) & & \rho_f | D_p \\
\updownarrow \text{ local Langlands} & & \updownarrow \\
\text{Weil-Deligne rep'n} & \xleftarrow{\text{forget filtration}} & D_{pst}(\rho_f|D_p) = \text{weakly adm. filtered } \varphi, N \text{ (Gal)-module} \\
\updownarrow & & \\
\varphi, N, \text{(Gal)-modules} & & [\, D_{st}(\rho_f|D_p) = \text{w.a fil. } \varphi, N\text{-module}\,]
\end{array}$$

Let me stick to the case where either (1) $p \nmid$ level of f (crystalline)
(2) $p \,\|$ level of f (semistable) & charact of f has conductor prime to p

In the last lecture

I unravelled things explicitly in cases 1 & 2.

In case 2, if you forget the filtration, you're sunk
∞ ly many rep'n.
$\longleftrightarrow$ $\mathcal{L}$-invariant of the situation $\in \overline{\mathbb{Q}}_p$

In case 1, a miracle occurs:
If $D = (\varphi, N)$-module, then $N = 0$.
& φ has e.val. the roots of $X^2 - a_p X + p^{k-1}\chi(p)$.
& If these roots are distinct, you can diagonalize φ
$$\varphi = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \quad \text{on } \overline{\mathbb{Q}}_p e_1 \oplus \overline{\mathbb{Q}}_p e_2$$
& up to isom. only

$\boxed{Ne_2 = e_1}$ 3 lines $\langle e_1 \rangle$, $\langle e_2 \rangle$, & any other $e_1 + \gamma e_2$

Furthermore, one isn't adm. & if $|a_p| > 0$ then 2 aren't adm.
So only one choice for filtration.

& $\pi_{f,p}$ & ρ determined by $\rho|D_p$

Rmk: This observation is essentially a coincidence:

It fails in alm. all other situations.

e.g. for Hilbert mod. forms

one replace $\mathbb{Q}_p$ by a fin. ext'n K of $\mathbb{Q}_p$ $K^\times/\mathbb{Q}_p^\times$

& if $K \neq \mathbb{Q}_p$, then always > 1 w.a. filtration

Exercise even fails for $GL_3(\mathbb{Q}_p)$: $(N=0)$

D: 3-dim'l / $\overline{\mathbb{Q}}_p$

$\varphi = \begin{pmatrix} \alpha & & \\ & \beta & \\ & & \gamma \end{pmatrix}$. The Hodge theory is a choice of a plane
e_1, e_2, e_3 $P \subseteq D$ & a line $L \subseteq P$.

If P & L are "in general position" w.r.t φ

then the filt'n is weakly admissible.

$\therefore$ w.a. filt'ns form an open in the flag variety.

(2-dim'l choice of possibilities for P
1-dim'l choice for L)

$\therefore$ $(P, L) \in$ 3-d. space

$e_1 \to \lambda e_1$
$e_2 \to \mu e_2$
$e_3 \to \gamma e_3$

The torus in $PGL_3(\overline{\mathbb{Q}}_p)$ acts on this space but this is only 2-dim'l only many orbits.

Back to the dictionary

Say f is a modular form of level N prime to p.

$T_p f = a_p f$, f has a char. χ

& assume $\chi(p) = 1$.

a_p unit $\rho_f = \begin{pmatrix} \text{cycl}^{k-1} & * \\ 0 & \cdot \end{pmatrix}$ Hodge Thy cannot read

Assume ~~that~~ $X^2 - a_p X + p^{k-1}$ has distinct roots.

$f \longrightarrow \pi_{f,p} \longrightarrow (\rho_f | D_p)^{ss}$

Let's now reduce $\rho_f^{ss} | D_p$ mod p. & we get $\overline{\rho}_{k, a_p} : G_{\mathbb{Q}_p} \to GL_2(\overline{\mathbb{F}}_p)$

a semi-simple mod p Galois rep'n

$\pi_{f,p} = \mathrm{Ind}(\chi_1, \chi_2)$ $\quad \left(\begin{array}{l} \chi_1|_{\mathbb{Z}_p^\times} = \chi_2|_{\mathbb{Z}_p^\times} = 1 \\ \chi_1(p) = \alpha \\ \chi_2(p) = \beta. \end{array} \right.$

Easy Cases $\qquad \overline{\mathbb{Q}}_p \cong \mathbb{C}$

- If $k=1$ then $\rho_f : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{C})$
  & ρ_f is unram. at p, $\rho_f(\mathrm{Frob}_p)$ is semi-~~simple~~ple
  (Hida) with e.val. the roots of $X^2 - a_p X + 1$.
- $|a_p| = 1$: $\rho_f|_{D_p}^{ss} = \mathrm{cyclo}^{k-1} \cdot \lambda(\alpha^{-1}) \oplus \lambda(\alpha)$
  $k \geq 2$
  where α = unit root of $X^2 - a_p X + p^{k-1}$
- For cases : $k \geq 2$ & $|a_p| < 1$.
  Set $v = v(a_p)$ $\quad (v(p) = 1)$

  If $2 \leq k \leq p+1$ & $v > 0$

  then $\overline{\rho}_{k,a_p}$ is irreducible & I can tell you what it is.

[Reminder of semi-simple Galois rep'ns

$\overline{\rho} : G_{\mathbb{Q}_p} \longrightarrow GL_2(\overline{\mathbb{F}}_p)$ with determinant ω^{k-1}.

ω = mod p cyclo char. $\qquad$ (from modular forms)

① Reducible ones, $\overline{\rho} = \chi_1 \oplus \chi_2$.

$\chi_1 = \omega^a \times \lambda(\alpha)$ $\quad \boxed{\text{arithmetic Frob.} \to \alpha}$ unramified char. $= \lambda(\alpha)$

$\chi_2 = \omega^{k-1}/\chi_1$ $\quad$ where $0 \leq a < p-1$ & $\alpha \in \overline{\mathbb{F}}_p^\times$

$$\begin{array}{c} G_{\mathbb{Q}_p}^{ab} = \mathbb{Z}_p^\times \times \hat{\mathbb{Z}} \\ \qquad\qquad \cup \\ \qquad\qquad \mathrm{Frob} \\ \downarrow \\ \mathbb{F}_p^\times \xrightarrow[\cong]{\omega} \mathbb{F}_p^\times \end{array}$$

② Irred. ones

wild inertia in $G_{\mathbb{Q}_p}$ is pro-p
& normal, so invariants under wild inertia are non-zero & an inv't submod.
$\therefore$ Wild inertia acts trivially

$\therefore$ tame inertia acts via a finite abelian gp of order prime to p.

$\therefore \bar{\rho}|I = \begin{pmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{pmatrix}$

& $\{\psi_1^p, \psi_2^p\} = \{\psi_1, \psi_2\}$ (Frobenius)

$\psi_1^p = \psi_1 \Rightarrow \bar{\rho}$ reducible. $\therefore \underset{\psi_2}{\overset{\psi_1^p}{\|}} \neq \psi_1, \psi_1^{p^2} = \psi_1$

$\therefore \quad \psi_1 = \omega_2^t$ $\quad \omega_2$ = fun. char. of niveau 2 (level)

where $t \in \mathbb{Z}$, $0 \leq t < p^2-1$

$p+1 \nmid t$,

$\psi_2 = \psi_1^p$.

& $\det \bar{\rho} = \omega^{k-1}$

$\Rightarrow t \equiv k-1 \mod p-1$

$\therefore \bar{\rho}$ is determined by t in this case

(call it $\boxed{I(t)}$)

$k \geq 2$ & $v > 0$

$0 \leq k \leq p+1$.

Then $\bar{\rho}_{k,a_p}$ is $I(k-1)$. For $k \leq p$ this is Fontaine-Laffaille

$\left(\begin{array}{ll} \mathrm{Fil}^i = D & j-i < p \\ \mathrm{Fil}^j = 0 & \end{array} \right]$

For $k = p$, there's a global proof due to Edixhoven

(There's now also a local proof Berger & Breuil).

Assume $p > 2$

$k = p+2$

If $v = 0$, then reducible $\quad$ char is $\lambda(\bar{a}_p)$ [Deligne]

(All rest is Berger-Breuil)

If $0 < v < 1$, then $\bar{\rho}_{k,a_p} = I(k-1-(p-1)) = I(2)$ $\quad \mathrm{Sym}^{k-2}(\bar{\mathbb{F}}_p^2)$ irreducible

If $v \geq 1$, then $\overline{\rho}_{k,a_p} = \omega \otimes$ unram semi-simple repln
with Frob_p e.vals
the roots of $X^2 - \left(\overline{\frac{a_p}{p}}\right) X + 1$.

Rmk: (Question) ? If $0 < v < 1$, then f lies on $\boxed{\text{a cpnt of}}$ eigencurve that also contains some form g. ?

$wt(g) = w(f) - (p-1)$

$v(a_p(g)) = v(a_p(f))$?

If $v \geq 1$,
then $\overline{f} = \Theta g$

$a_p(g) = \frac{a_p(f)}{p}$. g wt 1

"I.a.g." Θ. $S_k(\Gamma_1(N)\ \overline{\mathbb{F}_p})$ Katz

$\downarrow \qquad \downarrow$

"I.n.a.g." $S_{k+p+1}(\Gamma_1(N), \overline{\mathbb{F}_p})$

If f is an eigenform, then $\rho_{\Theta f} = \omega \otimes \rho_f$

If $p+3 \leq k \leq q$

then $0 < v < 1$. $\overline{\rho}_{k a_p} = I(k-1-(p-1))$
$= I(k-p)$

$v = 1$; if $\alpha = \left(\overline{\frac{a_p}{p}}\right) \times \underline{\underline{(k-1)}}$, then $\overline{\rho}_{k,a_p}$ is reducible,
& one character is $\omega \cdot \lambda(\alpha)$. ?

$\mathrm{Sym}^{k-2} \otimes \underline{\underline{\Pi_{f,p}}} \longleftrightarrow (\rho|_{D_p} \to D)$
} complete $\qquad$ } filtration.

$v > 1$: $I(k-1)$

General Thm of (Berger - Li - Zhu)

If $k \geq 2$ & $v > \left\lfloor \frac{k-2}{p-1} \right\rfloor$, then $\overline{\rho}_{k a_p} = I(k-1)$

unless $p+1 \mid k-1$, in which case it's $\omega^{\frac{k-1}{p+1}} \otimes$ unram s.s. repln with Frob. e.val. $\pm i$

Rk. So far answer has depended essentially only on $v(a_p)$

However, if $\pi_{f,p}$ = twist of Steinberg

then $\bar{\rho}_f|D_p$ is not determined by $\pi_{f,p}$.

& There are pts in the eigencurve :

$\longleftrightarrow$ newforms of level prime to p & have same slope

e.g. if $k=3$ & f is new @ p

& $v(a_p) = \frac{1}{2} = \frac{k-2}{2}$.

There's trouble in semi-stable case

Hence if $k = 3 + (p-1)\cdot p^N$ & $v(a_p) = \frac{1}{2}$,

$\bar{\rho}_{k,a_p}$ will **depend** on more than $v(a_p)$

If $k = 2p+1$ then

$V=0$: reducible. | char is $\lambda(\bar{a}_p)$

$0 < v < \frac{1}{2}$: irred.

$I(k-1-2(p-1)) = I(2)$

$v = \frac{1}{2}$: need to know more!

Need to know : W = val'n of $(\frac{a_p^2}{\chi(p)} + p)$

If $W < \frac{3}{2}$, then $\bar{\rho}_{k,a_p} = I(2)$

If $W \geq \frac{3}{2}$, then set $b = \dfrac{a_p^2+p}{2pa_p}$

$\bar{b} \neq 0$
$\Leftrightarrow W = \frac{3}{2}$

$\bar{\rho}_{k,a_p}$ is $\omega\otimes$ unram rep with Frob eval. the roots of $x^2 - \bar{b}x + 1$ $(\chi(p)=1$

If $v > \frac{1}{2}$, it's $I(k-1) = I(2p) \cong I(2)$

$v < \frac{1}{2}$. $I(k-1-2(p-1)) = I(2)$

$k = 2p+2$. Answer depends only on v! (computationally)

$0 < v < 1$: $I(k-1-2(p-1)) = I(3)$

$v = 1$: reducible, one char $\omega\lambda(a_p/p)$

$1 < v$: $I(k-1-(p-1)) = I(p+2)$

$2p+2 \le k \le 3p-1$

Answer depends only on v (& on $\overline{\left(\frac{a_p}{p^n}\right)}$ if $v=n$)

$k = 3p$: trouble @ $v = \frac{1}{2}$

$k = 3p+1$ " @ $v = 1$

$k = 3p+2$ " @ $v = 1\frac{1}{2}$

$p \ge 5$. $3p+3 \le k \le 4p-2$ only depends on v

$4p-1 \cdots 1+p+3$: trouble @ $v = \frac{1}{2}, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}$

$k = p^2$: trouble everywhere

$a_p^2 + p$

$2p+4 \le k \le 3p-1$ & $v = 2$

$\overline{\rho}_{k,a_p}$ reducible & one char is $\omega^2 \cdot \lambda\left(\frac{(k-1)(k-2)}{2}\overline{\left(\frac{a_p}{p^2}\right)}\right)$