May 2. 2006. Tuesday. 1:00pm Kevin Buzzard.
(20 th lecture)

Reminder of rep'n theoretic reformulation of "weight" part of Serres' conj : if $\rho : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(\overline{\mathbb{F}}_p)$

& $cond(\rho) = N$

& if $\mathbb{T} = \mathbb{Z}[T_\ell : \ell \nmid Np]$

then define $\lambda_\rho : \mathbb{T} \longrightarrow \overline{\mathbb{F}}_p$

$$\lambda_\rho(T_\ell) = trace\, \rho(Frob_\ell)$$

$\mathfrak{m} := \ker(\lambda_\rho)$ $\mathfrak{m}$ "knows ρ".

$\exists$ If ρ is modular of level N at $k \leq p+1$

$X_1(N;p) =$ descended modular curve traction of Γ s.t $\Gamma \backslash X_1(N;p) = X_1(N)$.

then $\mathfrak{m} \in$ support of $Hom_\Gamma(\sigma^*, H^1(X_1(N;p); \overline{\mathbb{F}}_p))$

$\Gamma = GL_2(\mathbb{Z}/p\mathbb{Z})$

$\sigma = Symm^{k-2}(\overline{\mathbb{F}}_p^2)$, $\Gamma \circlearrowleft \sigma$

Moral : Serre's conjecture has something to do with cohomology of $X_1(N;p)$ as a $GL_2(\mathbb{Z}/p\mathbb{Z})$-module.

Ash - Stevens Ash + Co-authors GL_3

Diamond emphasized this point of view when generalizing Serre's conj. to Hilbert modular forms. Herzig's Thesis rep'n of $Sl_2(\mathbb{C})$ $\longleftrightarrow$ integers

More recently, Serre's conj (& its generalizations) have been "encapsulated" as consequences of a more general bunch of conjectures essentially due to Breuil & Emerton.

Basic idea:

Serre's conj predicts the existence of non-trivial $Gl_2(\mathbb{Z}/p\mathbb{Z})$-equivalent homs from σ^* to $H^1(X_1(N,p), \overline{\mathbb{F}}_p)_m$

But why stop at level p?

Why not consider $\varinjlim H^1(X_1(N:p^m):\overline{\mathbb{F}}_p)_m$

+ this has an action of $Gl_2(\mathbb{Q}_p)$

& Concrete statements about this rep'n will enable us to read off the σ for which

$k \in Gl_2(\mathbb{A}_f)$
$\gamma \in Gl_2(\mathbb{Q}_p)$
$\gamma k \gamma^{-1}$

$$\mathrm{Hom}(\sigma^*, \underbrace{H^1(X_1(N:p), \overline{\mathbb{F}}_p)}_{})_m \text{ are non-zero.}$$

this is going to be the $K(1)$-invts of the big rep'n
$K(1) = \ker(Gl_2(\mathbb{Z}_p) \to Gl_2(\mathbb{F}_p))$

Dream (Breuil, Emerton)

if $\rho : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow Gl_2(\overline{\mathbb{F}}_p)$

is cts, odd, irreducible

then there should be a rep'n $\pi(\rho)$ of $Gl_2(\mathbb{Q}_p)$ on an ∞-dim'l $\overline{\mathbb{F}}_p$-vector sp.

s.t. $\pi(\rho)$ can be built globally. (union of coh. of modular curves, the m-torsion in $\varinjlim_n H^1(X_1(N\cdot p^n)\,\overline{\mathbb{F}}_p)$ + $GL_2(\mathbb{Q}_p)$-action.

but iso classes of $\pi(\rho)$ should depend only on $\rho|D_p$?

Then the weights that Serre predicts are precisely the σ's s.t. $\mathrm{Hom}_{GL_2(\mathbb{Z}/p\mathbb{Z})}(\sigma, \pi(\rho)^{K(1)})$ is non-zero

What would be nice now would be a complete list of all smooth irred adm. mod p rep'ns of $GL_2(\mathbb{Q}_p)$, plus for each such rep'n π, the list of all σ s.t $\mathrm{Hom}_{GL_2(\mathbb{Z}/p\mathbb{Z})}(\sigma, \pi^{K(1)})$ is non-zero.

Then we can guess def'n of $\pi(\rho)$.

In this lecture & the next

I'll explain how this "naive" approach (write down all ρ, all π & match 'em up)

has worked incredibly well for $GL_2(\mathbb{Q}_p)$

& has not yet worked at all for any other non-abelian reductive group.

GL_2(fin. ext'n of $\mathbb{Q}_p$) $\longleftrightarrow$ Hilbert MF's (Diamond)

$GL_3(\mathbb{Q}_p) \longleftrightarrow GL_3(\mathbb{Q})$ unitary gps || Ash et al. Herzig

$GL_2(\mathbb{Q}_{p^2})$ mod p rep'n (Christophe last lecture)

quaternion alg. ramified at p.

<u>Goal</u>: Write down as many irred. mod p rep'ns of $GL_n(F)$ where $n \geq 1$ & $F/\mathbb{Q}_p$ finite

Can we <u>write</u> them <u>all</u> down?

Before we start, there's a finite field version.

$\Gamma = GL_n(k)$, k: finite, $\# k = p^m$

E: alg. closed field of char p.

Let V be an irred rep'n of Γ on a f.d E-v. space.

Let B = upper triangular matrices in Γ.

Let T = diagonal ones

& let $U = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ = unipotent elts of B

Then $B = T \cdot U$

& There's a natural gp hom.

$$B \twoheadrightarrow T \quad \text{with kernel } U.$$

$$\begin{pmatrix} b_1 & & * \\ & b_2 & \\ & & \ddots \\ 0 & & & b_n \end{pmatrix} \to \begin{pmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_n \end{pmatrix}$$

V an irred rep'n of Γ.

Then $V^U := \{ v \in V : uv = v, \forall u \in U \}$

has an action of B as $U \triangleleft B$

& hence of $B/U = T$

T is abelian & $\# T$ is prime to p.

<u>Fact</u>: (Cartier, Lusztig)

$\dim V^U = 1$ & the associated character χ of T essentially determines V.

the map $\{\text{irred. rep'ns } V\} \longrightarrow \{\text{chars } \chi\}$

is surjective & fibers are typically of size 1

(size $>1 \iff \chi\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix} = \prod a_i^{r_i}$

& some of the r_i coincide)

Example : $\Gamma = GL_2(\mathbb{F}_p)$

The irred repns of Γ in char p are precisely

$Sym^g(\overline{\mathbb{F}}_p^2) \otimes \det^a \quad 0 \le a < p-1, \; 0 \le g \le p-1$.

Let me consider χ for $V = Sym^g(\overline{\mathbb{F}}_p^2)$

V can be thought of as the sp of hgs polys of

deg g in 2 variables X & Y

& $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f\right)(X, Y) = f(aX + cY, bX + dY)$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{F}_p)$

f is U-invariant $\iff f(X, X+Y) = f(X, Y)$

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

& this space is $\overline{\mathbb{F}}_p \cdot X^g$

$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in T$ acts on X^g via multi. by a^g.

& lo and behold we have recovered g from V^U.

except we can't distinguish the cases $g=0$ & $g=p-1$.

Now back to p-adic gps.

$F/\mathbb{Q}_p$ finite. units $\mathcal{O}$. residue field k. unif π

$G = GL_n(F)$

$K = GL_n(\mathcal{O})$ $\quad K \twoheadrightarrow \Gamma = GL_n(k)$. kernel is $K(1)$

How to construct irred reps of G on an E-v. space
(especially those that have a given rep'n V of Γ
in their $K(1)$-invariants)

<u>Idea</u>: Use induced rep'ns

Let V be an irreducible rep'n of Γ.

$K \twoheadrightarrow \Gamma$: V is an irred rep'n of K.

Let Z = center of G

$$Z = (Z \cap K) \times \langle \pi \rangle$$

$$\pi = \begin{pmatrix} \pi & & \\ & \pi & \\ & & \ddots \ \pi \end{pmatrix}$$

Extend V to a rep'n of $K \cdot Z$
by letting $\begin{pmatrix} \pi & & 0 \\ & \ddots & \\ 0 & & \pi \end{pmatrix}$ act trivially.

Define $c\text{-ind}_{KZ}^{G} V$ to be the "induced rep'n of G"
defined thus: $c\text{-ind}_{KZ}^{G}(V)$

$$\| $$

$$\Big\{ \text{fctns } f : G \longrightarrow V \quad \text{s.t.} \quad f(kg) = \underset{\underset{KZ\text{-action}}{\uparrow}}{k \cdot} f(g)$$

$$\forall g \in G,\ k \in K \cdot Z$$

& $\mathrm{supp}(f) \subseteq$ finite union of cosets $KZ \cdot g$ $\Big\}$
(compact mod center)

Define a G-action by $(g \cdot f)(h) = f(h \cdot g)$

($(h)(g \cdot f) = (hg) f$)

Frobenius Reciprocity :
if V is a rep'n of KZ & W is a rep'n of G,

then $\mathrm{Hom}_{KZ}(V, W|_{KZ}) = \mathrm{Hom}(\text{c-ind}_{KZ}^{G} V, W)$

Classically, when inducing from a Borel, result is irred.

These rep'ns $\text{c-ind}_{KZ}^{G} V$ are however far from irreducible

In fact, let's try & compute

$\boxed{\mathrm{End}_{E[G]}(\text{c-ind}_{KZ}^{G} V, \ \text{c-ind}_{KZ}^{G} V)} = \mathcal{H}(V) =$ "Hecke alg"

It'll be far from E.

By Frob. rec, this is $\mathrm{Hom}_{KZ}(V, (\text{c-ind}_{KZ}^{G} V))$

$\qquad\qquad\qquad\qquad \shortparallel$

maps $V \longrightarrow$ (maps $G \to V$ + axiom + finiteness condition)

+ axiom (KZ-inv)

$=$ maps $G \times V \longrightarrow V$ + finiteness + 2 axioms

$=$ maps $G \longrightarrow \mathrm{End}_E(V)$ + finiteness + 2 axioms

& one now unravels to check

$\mathcal{H}(V) = \{ f : G \longrightarrow \mathrm{End}_E(V)$ s.t.

convolution

$f(k_1 g k_2) = k_1 \circ f(g) \circ k_2 \ \forall k_1, k_2 \in KZ$

& s.t. $\mathrm{supp}(f) \subseteq$ fin. union of double cosets $KZgKZ \}$

<u>Exercise</u>: $\underset{GL_n(F)}{G} = \coprod_{g \in S} KZgKZ$

where $S = \left\{ \begin{pmatrix} 1 & & & \\ & \pi^{a_2} & & \\ & & \pi^{a_3} & \\ & & & \ddots \ \pi^{a_n} \end{pmatrix} : 0 \le a_2 \le a_3 \le \cdots \le a_n \right\}$

& $f \in \mathcal{H}(V)$ is determined by $f(g)$, $g \in S$,

all but fin. many of which are zero

In particular,

$$\mathcal{H}(V) = \bigoplus_{g \in S} \mathcal{H}(V)_g$$

$\uparrow$ elements of $\mathcal{H}(V)$ supported on $KZ\, g\, KZ$

Claim: (Kern. Schem. Herzig)

$\dim \mathcal{H}(V)_g = 1 \quad \forall g$

$n=2$ (Barthel – Livné)

Pf. for $n=2$

$S \ni \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ & $\begin{pmatrix} 1 & 0 \\ 0 & \pi^a \end{pmatrix}$, $a \geq 1$

If $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $f \in \mathcal{H}(V)_g$ is determined by its value on $g = id$: Say $f(id) = \alpha : V \to V$

Axiom implies that if $k \in KZ$, then $k\, id = id\, k$ & hence $k \circ \alpha = \alpha \circ k$ in $\mathrm{End}(V)$

V irred $\Rightarrow$ α is a scalar

$\Rightarrow \dim \mathcal{H}(V)_g = 1$ for $g = id$

If $g = \begin{pmatrix} 1 & 0 \\ 0 & \pi^a \end{pmatrix}$

& $f \in \mathcal{H}(V)_g$, then f is determined by $f(g) = \alpha : V \to V$

Axiom says: if $k_1, k_2 \in KZ$

& $k_1 g = g\, k_2$

then $k_1 \circ \alpha = \alpha \circ k_2$ as endom. of V.

One deduces that.

$$\begin{pmatrix} \lambda & \mu \\ 0 & \nu \end{pmatrix} \cdot \alpha = \alpha \cdot \begin{pmatrix} \lambda & 0 \\ \rho & \nu \end{pmatrix}$$

$$k_1 = \begin{pmatrix} \lambda & \mu \\ p^a\rho & \nu \end{pmatrix} \qquad k_2 = \begin{pmatrix} \lambda & p^a\mu \\ \rho & \nu \end{pmatrix}$$

$\cap$

$GL_2(k)$ $\quad \forall \lambda, \mu, \nu, \rho$ s.t. these are in $GL_2(k)$

General case

$$\boxed{\ell U \cdot \alpha = \alpha \cdot V \ell \qquad \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \pi & \\ & & & & \ddots \\ & & & & & \pi \end{pmatrix} = T}$$

levy $\quad$ unip radical

& hence α factors as

$$\begin{tikzcd} V \arrow[r, "\alpha"] \arrow[d] & V \\ V_{U^t} \arrow[r] & V^U \arrow[u, hook] \end{tikzcd}$$

V_{U^t}: 1-dim. $\qquad V^U$: $\leftarrow$ 1-dim

Hence $\dim \leq 1$
with equality $\iff V_{U^t} \cong V^U$
is T-equivariant,
which it is.

$\dim \mathcal{H}(V)_g = 1, \ \forall g \in S$

<u>Cor.</u> ($n=2$ ✓
$n \geq 3$ I've not checked details)

$\mathcal{H}(V) \cong E[T_1, \dots, T_{n-1}]$ $\quad$ poly. ring in $n-1$ variables

In particular, if $n \geq 2$ then $\mathrm{End}(c\text{-ind}_{KZ}^G V)$ is huge

Idea (Barthel – Livné)

If $\mathfrak{m} \subseteq \mathcal{H}(V)$ is a maximal ideal

consider the rep'n $\boxed{c\text{-ind}_{KZ}^G(V)/\mathfrak{m}}$

Is this irreducible?

For $n \geq 3$, **nothing** is known (to me) about this rep'n

– follow B-L.

<u>$n=2$.</u>

$\mathcal{H}(V) \cong E[T]$ & maximal ideals are $(T-\lambda)$, $\lambda \in E$.

Given V (irred. rep. of $GL_2(k)/E$)

& $\lambda \in E$

& $\chi : F^\times \to E^\times$ a cts char,

E: alg. closed, char p

F: fin. ext'n of $\mathbb{Q}_p$

define $\pi(V, \lambda, \chi) = \left(\dfrac{\text{c-ind}_{KZ}^{G} V}{T - \lambda} \right) \otimes (\chi \circ \det)$

an ∞-dim'l smooth rep'n of $G = GL_2(F)$

B-L proved that

$\pi(V, \lambda, \chi)$ was irreducible for $\lambda \neq 0$,

except in the case $\dim V = 1$ or $\dim V = \# k$ (biggest)

in which case $\pi(V, \lambda, \chi)$ may have two J-H factors, one of which is 1-dim.

& other of which is "Steinberg"

They failed to analyse $\underline{\lambda = 0 \text{ case}}$

Breuil proved

$\pi(V, 0, \chi)$ was irred. when $F = \mathbb{Q}_p$

& for $F = \mathbb{Q}_p$ one can now write down all smooth irreducible adm. reps of $GL_2(\mathbb{Q}_p)/E$.

$GL_n(\mathbb{Q}_p)$ $\rho : G_{\mathbb{Q}_p} \to GL_n(\overline{\mathbb{F}}_p)$ $\rho = \bigoplus_{i=1}^{n} \chi_i$

$\pi(\rho) \supseteq \prod PS$

$n!$ $n!$ weights.

$\text{soc}_K \left(I(\chi)^{K(1)} \right)$ $\underline{\underline{\text{irred}}}$

c-ind$_K$ $\updownarrow$ ind$_B$