LECTURE 1: (φ,Γ)-MODULES OVER THE ROBBA RING

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1. The Robba ring

Let[1] L be a finite extension of $\mathbb{Q}_p$, and let $\mathcal{R}_L$ be the Robba ring with coefficients in L, *i.e.* the ring of power series

$$f(z) = \sum_{n\in\mathbb{Z}} a_n (z-1)^n, \ \ a_n \in L$$

converging on some annulus of $\mathbb{C}_p$ of the form $r(f) \leq |z-1| < 1$, equipped with its natural L-algebra topology. It is a domain, but it is not noetherian.

Theorem 1 (Lazard, see e.g. [Berger1] prop. 4.12)**.**

(i) *Any finitely generated ideal of $\mathcal{R}_L$ is principal.*
(ii) *Any finite type submodule of $\mathcal{R}_L^n$ admits elementary divisors.*

Remark 1. Part (i) implies that finite type, torsion free, $\mathcal{R}_L$-modules are free. Part (ii) implies e.g. that if $M \subset N := \mathcal{R}_L^n$ is of finite type over $\mathcal{R}_L$, then the saturation of M in N (that is $M^{\mathrm{sat}} = \{x \in N, \exists f \neq 0 \in \mathcal{R}_L, fx \in M\}$) is finite type over $\mathcal{R}_L$ with the same rank as M.

The ring $\mathcal{R}_L$ is equipped with commuting, L-linear, continuous actions of φ and $\Gamma := \mathbb{Z}_p^*$ defined by

$$\varphi(f)(z) = f(z^p), \ \ \gamma(f)(z) = f(z^\gamma).$$

(note that here $z \in \mathbb{C}_p$ satisfies $|z-1| < 1$). Set

$$t := \log(z) := \sum_{n\geq 1} (-1)^{n+1} \frac{(z-1)^n}{n} \in \mathcal{R}_L.$$

Then $\varphi(t) = pt$ and $\gamma(t) = \gamma t$.

Lemma 1 ([Colmez2] rem. 4.4)**.** *The finitely generated ideals of $\mathcal{R}_L$ stable by φ and Γ are the $t^i\mathcal{R}_L$, $i \geq 0$ an integer.*

2. (φ,Γ)-modules over $\mathcal{R}_L$

Definition 1. A (φ,Γ)-module over $\mathcal{R}_L$ is a finite free $\mathcal{R}_L$-module D equipped with commuting, $\mathcal{R}_L$-semilinear, continuous[2] actions of φ and Γ, and such that $\mathcal{R}\varphi(D) = D$.

[1] Most of these notes have been extracted verbatim from the chapter 2 of [BelChe].

[2] It means that for any choice of a free basis $e = (e_i)_{i=1\ldots d}$ of D as $\mathcal{R}_L$-module, the matrix map $\gamma \mapsto M_e(\gamma) \in \mathrm{GL}_d(\mathcal{R}_L)$, defined by $\gamma(e_i) = M_e(\gamma)(e_i)$, is a continuous function on Γ. If $P \in \mathrm{GL}_d(\mathcal{R}_L)$, then $M_{P(e)}(\gamma) = \gamma(P)M_e(\gamma)P^{-1}$, hence it suffices to check it for a single basis.

Works of Fontaine, Cherbonnier-Colmez, and Kedlaya, allow to define a $\otimes$-equivalence D_{rig} between the category of L-representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and *étale* (φ,Γ)-modules over $\mathcal{R}_L$. By [Berger1, §3.4], $D_{\mathrm{rig}}(V)$ can be defined in Fontaine's style: there exists a topological ring B (denoted $B^{\dagger,\mathrm{rig}}$ there) equipped with actions of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and φ and such that $B^{H_p}=\mathcal{R}$, and

$$D_{\mathrm{rig}}(V) := (V\otimes_{\mathbb{Q}_p} B)^{H_p}.$$

Here, H_p is the kernel of the cyclotomic character $\chi:\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)\longrightarrow \mathbb{Z}_p^*$, inducing an isomorphism $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)/H_p \xrightarrow{\sim} \Gamma$.

Theorem 2. *(*[Colmez2, prop. 2.7]*) The functor D_{rig} induces an $\otimes$-equivalence of categories between finite dimensional, continuous, L-representations of* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ *and étale (φ,Γ)-modules over $\mathcal{R}_L$. We have* $\mathrm{rk}_L(V)=\mathrm{rk}_{\mathcal{R}_L}(D_{\mathrm{rig}}(V))$.

Remark 2. A (φ,Γ)-module is étale if its underlying φ-module has slope 0 in the sense of Kedlaya (see [Kedlaya, Theorem 6.10] or [Colmez2, §2.1]). Kedlaya defines some notion of slopes for φ-modules over $\mathcal{R}_L$ (such that $\varphi(M)\mathcal{R}_L=M$) and proves that any such module has a canonical filtration by isoclinic φ-submodules whose slopes are strictly increasing ([Kedlaya, Theorem 6.10]). In the (φ,Γ)-module situation, this φ filtration turns out to be stable by Γ (see [Berger2] part IV).

3. (φ,Γ)-modules of rank 1 and their extensions, following Colmez.

Let $\delta:\mathbb{Q}_p^*\longrightarrow L^*$ be a continuous character. Colmez defines in [Colmez2, §0.1], the (φ,Γ)-module $\mathcal{R}_L(\delta)$ which is $\mathcal{R}_L$ as $\mathcal{R}_L$-module but equipped with the $\mathcal{R}_L$-semilinear actions of φ and Γ defined by

$$\varphi(1):=\delta(p),\ \ \gamma(1):=\delta(\gamma),\forall\gamma\in\Gamma,$$

Recall that by class field theory the cyclotomic character χ extends uniquely to an isomorphism $\theta: W_{\mathbb{Q}_p}^{\mathrm{ab}}\xrightarrow{\sim}\mathbb{Q}_p^*$ sending the geometric Frobenius to p, where $W_{\mathbb{Q}_p}\subset\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is the Weil group of $\mathbb{Q}_p$. We may then view any δ as above as a continuous homomorphism $W_{\mathbb{Q}_p}\longrightarrow L^*$. Such a homomorphism extends continuously to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ iff $v(\delta(p))$ is zero, and in this case we see that

$$\mathcal{R}_L(\delta)=D_{\mathrm{rig}}(\delta\circ\theta).$$

Theorem 3. [Colmez2, Thm 0.2]

(i) *Any (φ,Γ)-module free of rank 1 over $\mathcal{R}_L$ is isomorphic to $\mathcal{R}_L(\delta)$ for a unique δ. Such a module is isocline of slope $v(\delta(p))$.*

(ii) *$Ext_{(\varphi,\Gamma)}(\mathcal{R}_L(\delta_2),\mathcal{R}_L(\delta_1))$ has L-dimension 1 except when $\delta_1\delta_2^{-1}=x^{-i}$ or $\chi.x^i$ for $i\geq 0$ an integer, in which case it has dimension 2.*

Here, $x:\mathbb{Q}_p^*\longrightarrow L^*$ is the inclusion, and $\chi=x|x|$ is the character such that $\chi(p)=1$ and $\chi_{|\Gamma}=x_{|\Gamma}$ is the natural inclusion. Colmez computes also Kedlaya's slopes of such extensions (see Rem. 0.3 of [Colmez2]). An important fact is that the extension can be étale (hence coming from a p-adic representation) even if the $\mathcal{R}_L(\delta_i)$'s are not. Some necessary conditions of etaleness are that $v(\delta_1(p))\geq 0$ and $v(\delta_1(p)\delta_2(p))=0$ (étalness of the determinant), these conditions are also sufficient in most cases (but see *loc. cit.*).

Definition 2. Let D be a (φ,Γ)-module of rank d over $\mathcal{R}_L$ and equipped with a strictly increasing filtration $(\mathrm{Fil}_i(D))_{i=0\ldots d}$:

$$\mathrm{Fil}_0(D) := \{0\} \subsetneq \mathrm{Fil}_1(D) \subsetneq \cdots \subsetneq \mathrm{Fil}_i(D) \subsetneq \cdots \subsetneq \mathrm{Fil}_{d-1}(D) \subsetneq \mathrm{Fil}_d(D) := D,$$

of (φ,Γ)-submodules which are free and direct summand as $\mathcal{R}_L$-modules. We call such a D a *triangular* (φ,Γ)-module over $\mathcal{R}_L$, and the filtration $\mathcal{T} := (\mathrm{Fil}_i(D))$ *a triangulation of D over $\mathcal{R}_L$.*

Following Colmez, we shall say that a (φ,Γ)-module which is free of rank d over $\mathcal{R}_L$ is *triangulable* if it can be equipped with a triangulation $\mathcal{T}$; we shall say that an L-representation V of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is *trianguline* if $D_{\mathrm{rig}}(V)$ is triangulable.

Let D be a triangular (φ,Γ)-module. By theorem. 3 (i), each

$$\mathrm{gr}_i(D) := \mathrm{Fil}_i(D)/\mathrm{Fil}_{i-1}(D),\ \ 1 \le i \le d,$$

is isomorphic to $\mathcal{R}_L(\delta_i)$ for some unique $\delta_i : W_{\mathbb{Q}_p} \longrightarrow L^*$. It makes then sense to define the *parameter of the triangulation* to be the continuous homomorphism

$$\delta := (\delta_i)_{i=1,\cdots,d} : \mathbb{Q}_p^* \longrightarrow (L^*)^d.$$

4. p-adic Hodge theory of (φ,Γ)-modules, following Berger.

Let D be a fixed (φ,Γ)-module. When $D = D_{\mathrm{rig}}(V)$ for some p-adic representation V, D uniquely determines V hence it makes sense to ask whether we can directly recover from D the usual Fontaine's functors of V. The answer is yes and achieved by Berger's work ([Berger1], [Berger2]). It turns out that it makes sense to define these Fontaine functors for any (φ,Γ)-module over $\mathcal{R}_L$ (i.e. not necessarily étale). In what follows, we may and do assume that $L = \mathbb{Q}_p$, $\mathcal{R} := \mathcal{R}_{\mathbb{Q}_p}$.

Let us introduce, for $r > 0 \in \mathbb{Q}$, the $\mathbb{Q}_p$-subalgebra

$$\mathcal{R}_r = \{f(z) \in \mathcal{R},\ f \text{ converges on the annulus } p^{-\frac{1}{r}} \le |z-1| < 1\}.$$

Note that $\mathcal{R}_r$ is stable by Γ, and that φ induces a map $\mathcal{R}_r \longrightarrow \mathcal{R}_{pr}$ when $r > \frac{p-1}{p}$ which is étale of degree p. The following lemma is [Berger2, thm 1.3.3]:

Lemma 2. *Let D be a (φ,Γ)-module over $\mathcal{R}$. There exists a $r(D) > \frac{p-1}{p}$ such that for each $r > r(D)$, there exists a unique finite free, Γ-stable, $\mathcal{R}_r$-submodule D_r of D such that $\mathcal{R} \otimes_{\mathcal{R}_r} D_r \xrightarrow{\sim} D$ and that $\mathcal{R}_{pr}D_r$ has a $\mathcal{R}_{pr}$-basis in $\varphi(D_r)$. In particular, for $r > r(D)$,*

(i) *for $s \ge r$, $D_s = \mathcal{R}_s D_r \xrightarrow{\sim} \mathcal{R}_s \otimes_{\mathcal{R}_r} D_r$,*
(ii) *φ induces an isomorphism $\mathcal{R}_{pr} \otimes_{\mathcal{R}_r,\varphi} D_r \xrightarrow{\sim} D_{pr} \xrightarrow{\sim} \mathcal{R}_{pr} \otimes_{\mathcal{R}_r} D_r$.*

If $n(r)$ is the smallest integer n such that $p^{n-1}(p-1) \ge r$, then for $n \ge n(r)$ the primitive p^n-th roots of unity lie in the annuli $p^{-\frac{1}{r}} < |z-1| < 1$ and t is a uniformizer at each of them so that we get by localization and completion at their underlying closed point a natural map

$$\mathcal{R}_r \longrightarrow K_n[[t]],\ n \ge n(r),\ r > r(D),$$

which is injective with t-adically dense image, where $K_n := \mathbb{Q}_p(\sqrt[p^n]{1})$. For any (φ,Γ)-module over $\mathcal{R}$, we can then form for $r > r(D)$ and $n \ge n(r)$ the space

$$D_r \otimes_{\mathcal{R}_r} K_n[[t]],$$

which is a $K_n[[t]]$-module free of rank $\mathrm{rk}_{\mathcal{R}}(D)$ equipped with a semi-linear continuous action of Γ. By Lemma 2 (i), this space does not depend on the choice of r such that $n \geq n(r)$. Moreover, for a fixed r, φ induces by the same lemma part (ii) $- \otimes_{\mathcal{R}_{pr}} K_{n+1}[[t]]$ a Γ-equivariant, $K_{n+1}[[t]]$-linear, isomorphism

$$(D_r \otimes_{\mathcal{R}_r} K_n[[t]]) \otimes_{t \mapsto pt} K_{n+1}[[t]] \longrightarrow D_r \otimes_{\mathcal{R}_r} K_{n+1}[[t]].$$

(Note that the map $\varphi : \mathcal{R}_r \longrightarrow \mathcal{R}_{pr}$ induces the inclusion $K_n[[t]] \longrightarrow K_{n+1}[[t]]$ such that $t \mapsto pt$.)

We use this to define functors $\mathcal{D}_{\mathrm{Sen}}(D)$ and $\mathcal{D}_{\mathrm{dR}}(D)$, as follows. Let $K_\infty = \bigcup_{n \geq 0} K_n$. For $n \geq n(r)$ and $r > r(D)$, we define a K_∞-vector space with a semi-linear action of Γ by setting

$$\mathcal{D}_{\mathrm{Sen}}(D) := (D_r \otimes_{\mathcal{R}_r} K_n) \otimes_{K_n} K_\infty.$$

By the discussion above, this space does not depend of the choice of n, r. In the same way, the $\mathbb{Q}_p$-vector spaces

$$\mathcal{D}_{\mathrm{dR}}(D) := (K_\infty \otimes_{K_n} K_n((t)) \otimes_{\mathcal{R}_r} D_r)^\Gamma,$$

$$\mathrm{Fil}^i(\mathcal{D}_{\mathrm{dR}}(D)) := (K_\infty \otimes_{K_n} t^i K_n[[t]] \otimes_{\mathcal{R}_r} D_r)^\Gamma \subset \mathcal{D}_{\mathrm{dR}}(D), \ \ \forall i \in \mathbb{Z},$$

are independent of $n \geq n(r)$ and $r > r(D)$. As $K_\infty((t))^\Gamma = \mathbb{Q}_p$, $\mathcal{D}_{\mathrm{dR}}(D)$ so defined is a finite dimensional $\mathbb{Q}_p$-vector-space whose dimension is less than $\mathrm{rk}_{\mathcal{R}}(D)$, and $(\mathrm{Fil}^i(\mathcal{D}_{\mathrm{dR}}(D)))_{i \in \mathbb{Z}}$ is a decreasing, exhausting, and saturated, filtration on $\mathcal{D}_{\mathrm{dR}}(D)$.

We end by the definition of $\mathcal{D}_{\mathrm{crys}}(D)$. Let

$$\mathcal{D}_{\mathrm{crys}}(D) := D[1/t]^\Gamma.$$

It has an action of $\mathbb{Q}_p[\varphi]$ induced by the one on $D[1/t]$. It has also a natural filtration defined as follows. Choose $r > r(D)$ and $n \geq n(r)$, there is a natural inclusion

$$\mathcal{D}_{\mathrm{crys}}(D) \longrightarrow \mathcal{D}_{\mathrm{dR}}(D)$$

and we denote by $(\varphi^n(\mathrm{Fil}^i(\mathcal{D}_{\mathrm{crys}}(D)))_{i \in \mathbb{Z}}$ the filtration induced from the one on $\mathcal{D}_{\mathrm{dR}}(D)$. By the analysis above, this defines a unique filtration $(\mathrm{Fil}^i(\mathcal{D}_{\mathrm{crys}}(V)))_{i \in \mathbb{Z}}$, independent of the above choices of n and r. We summarize some of Berger's results ([Berger1, thm. 0.2 and §5.3], [Berger2], [Colmez1, prop. 5.6])) in the following proposition.

Theorem 4. *Let V be a $\mathbb{Q}_p$-representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, and*

$$* \in \{crys, dR, Sen\}.$$

Then $\mathcal{D}_*(D_{\mathrm{rig}}(V))$ is canonically isomorphic to $D_*(V)$.

Definition 3. We will say that a (not necessarily étale) (φ, Γ)-module D over $\mathcal{R}$ is crystalline (resp. de Rham) if $\mathcal{D}_{\mathrm{crys}}(D)$ (resp. $\mathcal{D}_{\mathrm{dR}}(D)$) has rank $\mathrm{rk}_{\mathcal{R}}(D)$ over $\mathbb{Q}_p$. The Sen polynomial of D is the one of the semi-linear Γ-module $\mathcal{D}_{\mathrm{Sen}}(D)$.

Here is an example of application to triangular (φ, Γ)-modules. Let D be a triangular (φ, Γ)-module of rank d over $\mathcal{R}_L$, whose parameter is $(\delta_i)_{i=1,\ldots,d}$. Define the *weight* $\omega(\delta) \in L$ of any continuous character $\delta : \mathbb{Q}_p^* \longrightarrow L^*$ by the formula

$$\omega(\delta) := -\left(\frac{\partial \delta_{|\Gamma}}{\partial \gamma}\right)_{\gamma=1} = -\frac{\log(\delta(1+p^2))}{\log(1+p^2)} \ \in L.$$

Proposition 1. [BelChe, prop. 2.3.3, 2.3.4] *Let D be a triangular (φ, Γ)-module over $\mathcal{R}_L$ with parameter $(\delta_i)_{i=1,\dots,d}$.*

(i) *The Sen polynomial of $D_{\mathrm{Sen}}(D)$ is $\prod_{i=1}^{d}(T - \omega(\delta_i))$.*

(ii) *Assume that each $\omega(\delta_i) \in \mathbb{Z}$ and that the sequence $\omega(\delta_1), \omega(\delta_2), \dots, \omega(\delta_d)$ is strictly increasing. Then D is de Rham.*

References

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