Mar 8. 2006. Wed. Peter Schneider (Lecture I)

p-adic Banach spaces 1.

$\mathbb{Q}_p \subseteq L$ finite base field

K coefficient field

$\mathcal{G}_L := \mathrm{Gal}(\overline{L}/L)$

$$\begin{tikzcd}
\{1\} \arrow[d, dash, "\text{pro-}p\ \text{gp}"] & \overline{L} \arrow[d, dash] & \\
\text{tamely ram. gp}\ P_L \arrow[d, dash, "\prod_{\ell \neq p} \mathbb{Z}_\ell \cong\ \text{Kummer Theory}"'] & L^{tr} \arrow[d, dash] & \text{max. tamely ram} \\
\text{inertia gp}\ I_L \arrow[d, dash, "\phi_L^{\hat{\mathbb{Z}}} = \hat{\mathbb{Z}}"'] & L^{ur} \arrow[d, dash] & \text{max unram} \supseteq \text{all roots of unity of order prime to } p \\
\mathcal{G}_L & L &
\end{tikzcd}$$

pro-order prime to p

arithmetic Frob $x \mapsto x^q$

Local class field theory

$$\begin{tikzcd}
1 \arrow[r] & O_L^\times \arrow[r] \arrow[d, "\cong"] & L^\times \arrow[r] \arrow[d, "\mathrm{rec}"] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
1 \arrow[r] & \mathrm{im}(I_L^{ab}) \arrow[r] & \mathcal{G}_L^{ab} \arrow[r] & \phi_L^{\hat{\mathbb{Z}}} \arrow[r] & 1
\end{tikzcd}$$

$-1 \mapsto \phi_L^{-1}$

rec : dense image

Weil gp

$$\begin{tikzcd}
1 \arrow[r] & I_L \arrow[r] \arrow[d, equal] & W_L \arrow[r] \arrow[d, phantom, "\cap"] & \phi_L^{\mathbb{Z}} \arrow[d, phantom, "\cap"] & \\
1 \arrow[r] & I_L \arrow[r] & \mathcal{G}_L \arrow[r] & \phi_L^{\hat{\mathbb{Z}}} \arrow[r] & 1
\end{tikzcd}$$

W_L is retopologized by declaring I_L to be open.

- Idea of local Langlands is to understand G_L through its representation.
- <u>Reformulation</u> of LCT.

$$\begin{matrix} \text{1-dim. discrete} \\ \text{rep'n of } W_L \end{matrix} \quad \overset{\sim}{\longleftrightarrow} \quad \begin{matrix} \text{irreducible smooth} \\ \text{rep'n of } L^\times = GL_1(L) \end{matrix}$$

- <u>Vague formulation</u> of L.L.

: There are bijections for any $n \geq 1$

$$\begin{matrix} \text{n-dim. discrete} \\ \text{rep'n of } W_L \end{matrix} \quad \overset{\sim}{\longleftrightarrow} \quad \begin{matrix} \text{irreducible smooth} \\ \text{rep'n of } GL_n(L) \end{matrix}$$

Which representations are we talking about?

Fix an alg. closed field K of char 0.

<u>Right hand side</u> :

<u>Def</u>. A smooth rep'n of $GL_n(L)$ is a linear action on a K-vector space such that all stabilizers

$$G_v := \{ g \in G \; : \; g.v = v \}, \text{ for any } v \in V$$

are <u>open</u>.

"irreducible" means the obvious

<u>Ex</u> $GL_2(L)$, $V :=$ locally constant ftns $\mathbb{P}^1(L) \longrightarrow K$

Fact: $\{0\} \subseteq$ const ftns $\subseteq V$ is a J-H series.

$\uparrow$ Steinberg (inf.-dim.)

Left Hand Side:

"discrete" = "smooth"

Attempt 1: n-dim discrete rep'n of W_L is a conti. homo. $\rho : W_L \longrightarrow \underset{\text{discrete top.}}{GL_n(K)}$

$\ker(\rho)$ is open $\Rightarrow$ there is an open subgp $\mathcal{U} \subseteq I_L$ of finite index s.t. $\rho|_{\mathcal{U}} = 1$.

These are not enough objects, e.g. 1 and Steinberg cannot be distinguished.

Attempt 1'. ($K = \mathbb{C}$) $\rho : W_L \dashrightarrow GL_n(\mathbb{C})$ conti. for natural topology

fact. $\rho(I_L)$ is necessarily finite $\Rightarrow$ Same objects as before

Attempt 2 (Galois rep'ns arise through l-adic cohomology of arithmetic varieties)

Consider $K = \overline{\mathbb{Q}_\ell}$ for $\ell \neq p$.

and $\rho : W_L \dashrightarrow GL_n(\overline{\mathbb{Q}_\ell})$ conti w.r.t the natural topology

(P_L: pro-p $\Rightarrow \exists\, \mathcal{U} \subseteq P_L$ open st. $\rho|_{\mathcal{U}} = 1$)

- Abstract monodromy theorem (Grothendieck)

- There is an equivalence of categories

$$\left(\begin{array}{l}\text{Conti homomorphism} \\ \rho : W_L \to GL_n(\overline{\mathbb{Q}_\ell}) \\ \quad (\text{natural top.})\end{array}\right) \xrightarrow{\sim} \left(\begin{array}{l}\text{pairs } (\sigma, N) \text{ where} \\ - \sigma : W_L \to GL_n(\overline{\mathbb{Q}_\ell}) \text{ discrete topology} \\ - N \in M_n(\overline{\mathbb{Q}_\ell}) \text{ a nilpotent matrix} \\ \quad \text{s.t. } \sigma(w) \cdot N = |w| \cdot N\, \sigma(w) \\ \quad \text{for all } w \in W_L\end{array}\right)$$

→ rep'n of the Weil-Deligne group

"after rec^{-1}"

$|\,| : W_L \to \overline{\mathbb{Q}}^\times$ conti st $|w| = 1$, $w \in I_L$, $|\varphi_L^{-1}| = q^{-1}$

- Thm about L.L (Harris & Taylor, Henniart)

There are "canonical" bijections, for every $n \geq 1$, between isomorphism classes of

n-dim pairs (σ, N) as above with σ being semi-simple	and	irreducible smooth rep'n of $GL_n(L)$

- Our example.

$$\left(\sigma(w) = \begin{pmatrix} 1 & 0 \\ 0 & |w| \end{pmatrix}, N = 0 \right) \longleftrightarrow 1$$

$$\left(\sigma(w) = \begin{pmatrix} 1 & 0 \\ 0 & |w| \end{pmatrix}, N \neq 0 \right) \longleftrightarrow \text{Steinberg}$$

$\boxed{\text{Attempt 3}}$: What about $\ell = p$?

No visible restriction on ρ any more.

Means we have a lot more elements

$\rightsquigarrow$ Should make right hand side considerably larger as well.

$\rightsquigarrow$ Consider Banach space rep'n of $GL_n(L)$
(next time!)

- The Fontaine functor

(ρ, E) is a conti rep'n $\rho : W_L \to GL(E)$ where E is an n-dim K-vector space.

(From now on: $K/\mathbb{Q}_p$ is finite)

$P_0 := \widehat{\mathbb{Q}_p^{ur}}$, carries absolute Frobenius σ.

B_{st} := A specific P_0-algebra equipped with

- a semi-linear $\mathcal{G}_L$-action
- a σ-linear injective Frob. endomorphism $\phi : B_{st} \longrightarrow B_{st}$ commuting with Galois.
- a linear derivation $N : B_{st} \longrightarrow B_{st}$ commuting with Galois and satisfying $N \circ \phi = p \cdot \phi \cdot N$.

Define the functor

$$\mathrm{Fon}(\rho, E) = \bigcup_{\substack{H \subseteq I_L \\ \text{open}}} \left(B_{st} \otimes_{\mathbb{Q}_p} E\right)^H$$

$\mathrm{Fon}(\rho, E)$ is a discrete W_L-rep'n semi-linear for P_0

Define $\phi := \phi \otimes id_V$

$N := N \otimes id_V$

Define $\rho(w) := w \cdot \phi^{-\log_p |w|_L}$; a linear discrete action

$\rightsquigarrow$ (ρ, N) is a Weil-Deligne gp rep'n over $P_0 \otimes_{\mathbb{Q}_p} K$.