

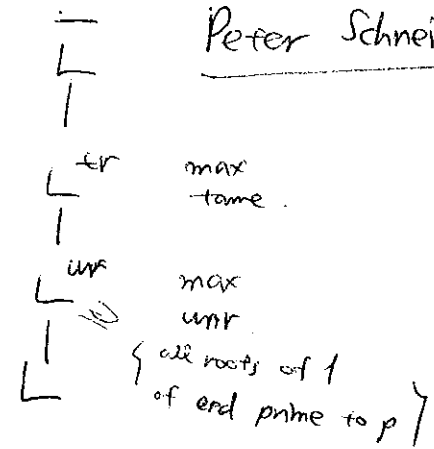
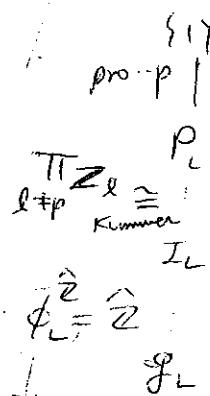
2006.3.8.

p-adic Ban. Space LLC.

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$\mathbb{Q}_p \subseteq L$ fin. base field
 K coeff. field.

$\mathcal{G}_L := \text{Gal}(\bar{L}/L)$



LCFT

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \mathcal{O}_L^X & \longrightarrow & L^X & \longrightarrow & \mathbb{Z} \longrightarrow 1 \\
 & & \cong \downarrow & & \downarrow \text{rec} & & \downarrow \phi_L^{-1} \\
 1 & \longrightarrow & \text{im}(\mathcal{I}_L^{\text{ab}}) & \longrightarrow & \mathcal{G}_L^{\text{ab}} & \longrightarrow & \phi_L^{\hat{\mathbb{Z}}} \longrightarrow 1 \\
 & & \text{dense} & & \text{image} & &
 \end{array}$$

$$\begin{array}{ccccccc}
 \text{weil sp} & & \mathcal{I}_L & \longrightarrow & W_L & \longrightarrow & \phi_L^{\mathbb{Z}} \\
 & & \parallel & & \cap & & \cap \\
 1 & \longrightarrow & \mathcal{I}_L & \longrightarrow & \mathcal{G}_L & \longrightarrow & \phi_L^{\hat{\mathbb{Z}}} \longrightarrow 1.
 \end{array}$$

W_L is re-topologized by declaring \mathcal{I}_L is open.

idea of LLC:

understand \mathcal{G}_L thru its reps, reformulation of LCFT.

$$\left\{ \begin{array}{l} \text{1-dim'l discrete} \\ \text{rep of } W_L \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{l} \text{inv. sm.} \\ \text{rep of } L^X = \text{GL}_1(L) \end{array} \right\}$$

vague formulation of LLC

\exists bijections, $\forall n \geq 1$.

$$\left\{ \begin{array}{l} \text{n-dim'l discrete} \\ \text{rep. of } W_L \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{l} \text{inv. sm.} \\ \text{rep of } \text{GL}_n(L) \end{array} \right\}$$

Which reps are we talking about?

Taylor asked.

RHS Fix $K = \overline{K}$ of char 0, $(K \cong \mathbb{C})$

Def A smooth rep. of $GL_n(L)$ is a lin action on a K -v.sp. V

s.t. $G_v := \{g \in G : gv = v\}$, $\forall v \in V$, are open.

(or, cont w.r.t. disc. top on V).

Ex $GL_n(L)$ acts as left-transl.

$\tilde{V} := \text{loc. const. fns } IP^1(L) \rightarrow K$.

Fact $\{0\} \subseteq \{\text{const. fns}\} \subseteq V$

is a JH-series.

Steinberg rep.

LHS "discrete" = "smooth".

attempt 1

n -dim. rep of W_L is a cont. hom.

$$\rho: W_L \rightarrow GL_n(K)$$

disc. top.

$\ker(\rho)$ is open $\Rightarrow \exists$ an open subgrp $U \subseteq I_L$ of fin index
s.t. $\rho|_U = 1$.

* these are not enough, e.g. 1 and I_L cannot be distinguished.

attempt 1'

$$\rho: W_n \rightarrow GL_n(\mathbb{C})$$

cont. for natural top.

fact $\rho(I_L)$ is necessarily finite

\Rightarrow same objs as before.

attempt 2 (Gal repn arise thm
 l -adic coh. of arith var)

Consider $K = \overline{\mathbb{Q}_l}$ for $l \neq p$.

and $\rho: W_L \rightarrow GL_n(\overline{\mathbb{Q}_l})$. cont wrt the natural top.
 (l -adic)

ρ : prop. $\Rightarrow \exists U \subseteq \rho$ open s.t. $\rho|_U = 1$.

Abstract monodromy thm. (Groth).

\exists an equiv. of catef

$\left\{ \begin{array}{l} \text{cont. hom.} \\ \rho: W_L \rightarrow GL_n(\overline{\mathbb{Q}_l}) \end{array} \right\}^{\text{nat. top.}} \xrightarrow{\sim} \left\{ \begin{array}{l} \text{pairs } (\theta, N) \text{ where} \\ - \theta: W_L \rightarrow GL_n(\overline{\mathbb{Q}_l}) \text{ disc top.} \\ - N \in M_n(\overline{\mathbb{Q}_l}) \text{ a nilp. matrix s.t.} \\ \theta(w) \cdot N = (|w| \cdot N) \cdot \theta(w) \quad \forall w \in W \\ \text{(rep of the WD sp)} \\ \text{val}(\text{rec}(w)). \end{array} \right.$

Thm about LLC (HT, Henniart).

\exists "canonical" bij, $\forall n \geq 1$, bet. isom. classes of

$\left\{ \begin{array}{l} n\text{-dim pairs } (\theta, N) \\ \text{as above w/} \\ \theta \text{ being s.s.} \end{array} \right\}$ and $\left\{ \begin{array}{l} \text{ired. sm.} \\ \text{rep of } GL_n(L) \end{array} \right\}$

our example

$$\left(\theta(w) = \begin{pmatrix} 1 & 0 \\ 0 & |w| \end{pmatrix}, N=0 \right) \longleftarrow |$$

$$\left(\theta(w) = \begin{pmatrix} 1 & 0 \\ 0 & |w| \end{pmatrix}, N \neq 0 \right) \longleftarrow \text{St.}$$

N comes from the action of top. gen. of I_L/P_L
 (same inertia)

attempt 3 what about $l=p$?

No visible restr. on p anymore!

→ means we have many more obj's.

→ should make RHS considerably larger as well.

→ consider Banach space rep of $GL_n(L)$ (next time).

The Fontaine fer

motiv

There should be connection bet
 p -adic LCC and usual LCC.

(ρ, E) is a cont rep. $\rho: W_L \rightarrow GL(E)$.

where E is an n -dim. K -v. sp.

(From now on, K/\mathbb{Q}_p is fin)

$$P_0 := \hat{\mathbb{Q}}_p^{\text{ur}} \curvearrowright \sigma \text{ Frob}$$

$B_{st} :=$ a specific P_0 -alp. equipped w/

- a semilin. G_L -action.

- a σ -lin. bij Frob endom $\phi: B_{st} \rightarrow B_{st}$.

- a lin. derivation $N: B_{st} \rightarrow B_{st}$ commuting w/ Gal.
and satisfying $N \circ \phi = p \cdot \phi \circ N$.

one more
str. is
missing

Define the fer

$$\text{Fon}(\rho, E) := \bigcup_{\substack{H \subseteq \mathbb{Z}_L \\ \text{open}}} (B_{st} \otimes_{\mathbb{Q}_p} E)^H$$

* is a discrete W_L -rep,
semilin. for P_0

$$* \phi := \phi \otimes \text{id}_E$$

$$* N := N \otimes \text{id}_E$$

define $\rho(w) = w \circ \phi^{-\langle w, w \rangle_L}$ is a lin. disc. action.

→ (ρ, N) is a WD-rep. over $P_0 \otimes_{\mathbb{Q}_p} K$.

(interesting obj's
descend to K but...)