2006.3.8. p-adic Ban. space. LLC.

Peter Schneider

$\mathbb{Q}_p \subseteq L$ fin. base field

K coeff. field.

$\mathcal{G}_L := \mathrm{Gal}(\bar{L}/L)$.

$$\begin{array}{lll} & \bar{L} & \\ \text{pro-}p \ \{1\} \ P_L & | & \\ & L^{tr} & \text{max tame} \\ \prod_{\ell \neq p} \mathbb{Z}_\ell \underset{\text{Kummer}}{\cong} \ I_L & | & \\ & L^{unr} & \text{max unr} = \{\text{all roots of 1 of ord prime to } p\} \\ \phi_L^{\hat{\mathbb{Z}}} = \hat{\mathbb{Z}} \ \ \mathcal{G}_L & | & \\ & L & \end{array}$$

arith Frob. $x \mapsto x^q$

LCFT

$$\begin{array}{ccccccccc} 1 & \to & \mathcal{O}_L^\times & \to & L^\times & \to & \mathbb{Z} & \to & 1. \\ & & \cong \downarrow & & \downarrow \mathrm{rec} & & \downarrow \phi_L^{-1} & & \\ 1 & \to & \mathrm{im}(I_L^{ab}) & \to & \mathcal{G}_L^{ab} & \to & \phi^{\hat{\mathbb{Z}}} & \to & 1 \end{array}$$

dense image

Weil gp

$$\begin{array}{ccccccccc} 1 & \to & I_L & \to & W_L & \to & \phi_L^{\mathbb{Z}} & & \\ & & \| & & \cap & & \cap & & \\ 1 & \to & I_L & \to & \mathcal{G}_L & \to & \phi_L^{\hat{\mathbb{Z}}} & \to & 1. \end{array}$$

W_L is retopologized by declaring I_L is open.

idea of LLC:

understand $\mathcal{G}_L$ thru its reps, reformulation of LCFT.

$$\left\{ \begin{array}{c} \text{(1-dim'l) discrete} \\ \text{rep of } W_L \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{irr. sm.} \\ \text{rep of } L^\times = GL_1(L) \end{array} \right\}$$

vague formulation of LLC

$\exists$ bijections, $\forall n \geq 1$,

$$\left\{ \begin{array}{c} n\text{-dim'l discrete} \\ \text{rep. of } W_L \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \text{irred sm.} \\ \text{rep of } GL_n(L) \end{array} \right\}$$

· Which reps are we talking about? Taylor asked.

RHS Fix $K = \overline{K}$ of char 0, $\boxed{K \cong \mathbb{C}}$

Def A smooth rep. of $GL_n(L)$ is a lin action on a K-v.sp. V

s.t. $G_v := \{g \in G : gv = v\}$, $\forall v \in V$, are open.

(or, cont wrt disc. top on V)

Ex $GL_2(L)$ acts as left transl.

$V :=$ loc. const. fcns $\mathbb{P}^1(L) \to K$.

Fact $\{0\} \subseteq \{$const. fcns$\} \subseteq V$

is a JH-series. └ Steinberg. rep.

LHS "discrete" = "smooth"

attempt 1

n-dim. rep of W_L is a cont. hom.

$$\rho : W_L \longrightarrow \underset{\text{disc. top}}{GL_n(K)}$$

$\ker(\rho)$ is open $\Rightarrow$ $\exists$ an open subgp $U \subseteq I_L$ of fin index

s.t. $\rho|U = 1$.

* these are not enough, e.g. 1 and St cannot be distinguished

attempt 1' $\rho : W_n \longrightarrow GL_n(\mathbb{C})$

cont. for natural top.

fact $\rho(I_L)$ is necessarily finite

$\Rightarrow$ same objs as before.

attempt 2 (Gal repn arise thru ℓ-adic coh. of arith var).

Consider $K = \overline{\mathbb{Q}_\ell}$ for $\ell \neq p$

and $\rho : W_L \to GL_n(\overline{\mathbb{Q}_\ell})$ cont wrt the natural top. (ℓ-adic)

P : pro-p. $\Rightarrow \exists\, U \subseteq P$ open s.t. $\rho|U = 1$.

Abstract monodromy thm. (Groth).

$\exists$ an equiv. of categ

$$\left\{ \begin{array}{l} \text{cont. hom.} \\ \rho : W_L \to GL_n(\overline{\mathbb{Q}_\ell})^{\text{nat. top}} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{l} \text{pairs } (\sigma, N) \text{ where} \\ - \sigma : W_L \to GL_n(\overline{\mathbb{Q}_\ell})^{\text{disc top.}} \\ - N \in M_n(\overline{\mathbb{Q}_\ell}) \text{ a nilp. matrix s} \\ \quad \sigma(w) \circ N = |w| \cdot N \circ \sigma(w) \quad \forall w \in W \\ (\text{rep of the WD gp}) \end{array} \right.$$

val (rec(w)).

Thm about LLC (HT, Henniart).

$\exists$ "canonical" bij, $\forall n \geq 1$, bet. isom. classes of

$$\left\{ \begin{array}{l} n\text{-dim pairs } (\sigma, N) \\ \text{as above w/} \\ \sigma \text{ being s.s.} \end{array} \right\} \text{ and } \left\{ \begin{array}{l} \text{irred. sm.} \\ \text{rep of } GL_n(L) \end{array} \right\}.$$

our example

$$\left(\sigma(w) = \begin{pmatrix} 1 & 0 \\ 0 & |w| \end{pmatrix}, N = 0 \right) \longleftarrow 1.$$

$$\left(\sigma(w) = \begin{pmatrix} 1 & 0 \\ 0 & |w| \end{pmatrix}, N \neq 0 \right) \longleftarrow St.$$

N comes from the action of top. gen. of I_L / P_L (tame inertia) attempt 3 what about $l = p$?

No visible restr. on ρ anymore!

$\Rightarrow$ means we have many more obj's.

$\rightsquigarrow$ Should make RHS considerably larger as well.

$\rightsquigarrow$ consider Banach space rep of $GL_n(L)$ (next time).

The Fontaine fctr

motiv

There should be connection bet p-adic LLC and usual LLC.

(ρ, E) is a cont rep. $\rho : W_L \to GL(E)$.

where E is an n-dim. K-v. sp.

(From now on, $K/\mathbb{Q}_p$ is fin)

$P_0 := \hat{\mathbb{Q}}_p^{ur} \circlearrowleft \sigma$ Frob

B_{st} := a specific P_0-alg. equipped w/

- a semilin. $\mathcal{G}_L$-action.
- a σ-lin. bij Frob endom $\phi : B_{st} \to B_{st}$
- a lin. derivation $N : B_{st} \to B_{st}$ commuting w/ Gal and satisfying $N \circ \phi = p \cdot \phi \circ N$.

one more str. is missing

Define the fctr

$$\mathrm{Fon}(\rho, E) := \bigcup_{\substack{H \subseteq I_L \\ \text{open}}} (B_{st} \otimes_{\mathbb{Q}_p} E)^H .$$

* is a discrete W_L-rep, semilin. for P_0
* $\phi := \phi \otimes id_E$
* $N := N \otimes id_E$

Define $\rho(w) = w \circ \phi^{-\log_p |w|_L}$ is a lin. disc. action.

$\rightsquigarrow$ (ρ, N) is a WD-rep. over $P_0 \otimes_{\mathbb{Q}_p} K$.

(interesting obj's descend to K but ...)