Mar 10. 2006. Friday. Peter Schneider (lecture 2)
— p-adic Banach space rep'ns. 2

$K/\mathbb{Q}_p$: finite (Coefficient field)

V : vector spaces always over K.

Admissible smooth G-rep'n V.

: For any cpt open subgp $U \subseteq G$, the fixed vector V^U is finite dimensional.

(Fact. (Bernstein) : irred & smooth $\Rightarrow$ admissible)

• Banach space representations

G is any p-adic Lie gp.
(e.g. $G = GL_n(L)$ or any open subgp therein)

Def. A norm on a K-vector space V is a function
$\| \ \| : V \to \mathbb{R}_{\geq 0}$ s.t.

(i) $\|v + v'\| \leq \max(\|v\|, \|v'\|)$

(ii) $\|a v\| = |a| \|v\|$, $a \in K$

(iii) $\|v\| = 0 \Rightarrow v = 0$

gives a metric and hence a topology on V.

Def A K-Banach space is a topological K-v. sp whose topology can be defined by a norm and which is complete.

Def. A Banach rep'n V of G is a linear action of G on V s.t. the map $G \times V \to V$ is continuous.

$\mathrm{Ban}_K(G)$:= Category of these.

This is unreasonably big category! (non-abelian category)
$\exists$ injective map $V_0 \hookrightarrow V_1$ with dense image but not surjective.
both V_0 and V_1 are topologically irreducible.
(no closed proper G-inv space)

Ex.: (Diarra)

$G = \mathbb{Z}_p$. Take $z \in \mathbb{C}_p = \widehat{\overline{\mathbb{Q}}_p}$ s.t. $|z| < 1$

Let V_z be the smallest closed subfields of $\mathbb{C}_p$ containing z.

Then V_z is a $\mathbb{Q}_p$-Banach space

Let G act by $a \cdot v := (1+z)^a v$

Fact: if z is transcendental over $\mathbb{Q}_p$, then V_z is infinite-dim and is top. irreducible as G-rep'n.

Goal: Construct a "reasonable" subcategory of "admissible" Banach rep'n which is still rich enough.

Rmks: ① V is a Banach space with a norm $\|\cdot\|$.

$L = \{ v \in V : \|v\| \le 1 \}$ is an $\mathcal{O}_K$-submodule such that

* $K \otimes_{\mathcal{O}_K} L = V$ ("lattice") → doesn't require freeness as $\mathcal{O}_K$-module in this lecture
* L is open in V.
* L is bounded in V.

Vice versa: let $L \subseteq V$ be a bdd open lattice

then $\|v\|' := \inf_{\substack{v \in a \cdot L \\ a \in K}} |a|$ is a defining norm for V.

($\|\ \|$, $\|\cdot\|'$ are equivalent.)

② In general we do not find a defining norm which is preserved by G.

But, we do for any compact subgp $H \subseteq G$.

$\boxed{\text{Key Definition}}$:

A Banach rep'n V of G is called admissible, if there is a cpt open subgp $H \leq G$ and an H-invariant bdd open lattice $L \subset V$ such that the following finiteness conditions hold:

(*) for any open subgp $U \subseteq H$, the U-fixed elements $(V/L)^U$ are of cofinite type over $\mathcal{O}_K$.

Rmk: ① "cofinite type" means $(K/\mathcal{O}_K)^m \oplus$ finite

② if V is admissible, then for any cpt open subgp $H \subseteq G$, we can find an H-inv bdd open lattice s.t the finite conditions $(*)$ hold

③ Suppose V is admissible with H and L as in the def'n. V reduces to $\bar{L} = L/\pi_K L$ is a smooth H-rep'n over $\mathcal{O}_K/\pi_K$ which is admissible-smooth.

Define $Ban_K^a(G)$:= Category of all admissible ones.

$\boxed{\text{Thm}}$ (S. / Teitelbaum)

i. $Ban_K^a(G)$ is an abelian category

ii. All rep'ns in $Ban_K^a(G)$ are <u>strict</u> with <u>closed image</u>

(Quotient topology from the source = Subsp. topology from the target sp)

$\pi : G \xrightarrow[\text{closed}]{} Aut(V)$

$\pi(G)$ has two topolog (quot top, subsp top)

<u>Strict</u> means that they coincide

Strategy of Proof:

May assume that G is <u>compact</u>.

$C(G)$:= all conti. functions $G \to K$

Banach rep'n for sup-norm.
(left or right translation.)

$$\delta_g \in D^c(G, K) := \text{conti. dual of } C(G)$$

← weak topology (but this lecture consider it as a purely alg object)

$$\begin{matrix} \delta_g & \in & D^c(G,K) \\ \uparrow & & \cup \\ g & \in & G \end{matrix}$$

is in fact a K-algebra

s.t. $D^c(G, K) \cong K \otimes_{\mathcal{O}_K} \mathcal{O}_K[[G]]$

<u>Fact 1</u> : $\mathcal{O}_K[[G]]$ is <u>noetherian</u> (Lazard)

p-adic cpt Lie gp.

<u>Fact 2</u> : $Ban_K^a(G) \longrightarrow Mod_{f.g.}(D^c(G,K))$ ← purely algebraic category

$$V \longmapsto V' := \text{continuous dual}$$

is an <u>anti-equivalence</u> of categories.

<u>Rmk</u>. Taking dual V' looks natural.

($C(G)_{\circlearrowleft G \text{ (left translation)}} \longrightarrow D^c(G,K) = C(G)'$)

Continuous Principal series

$G = GL_n(\mathbb{Q}_p)$

$\cup$

$P :=$ lower triangular matrices

$\chi : P \longrightarrow K$: conti. character

$$\mathrm{Ind}_P^G(\chi) := \{ f : G \longrightarrow K \text{ conti} : \quad f(g \cdot p) = \chi(p)^{-n} \cdot f(g) \quad \forall g \in G,\ p \in P \}$$

G acts by left translation

Iwasawa decomp: $G = G_0 \cdot P$ with $G_0 := GL_n(\mathbb{Z}_p)$

Take sup-norm over $G_0 \rightsquigarrow \mathrm{Ind}_P^G(\chi)$ is an admissible Banach rep'n of G

$$\mathrm{Ind}_P^G(\chi) = \mathrm{Ind}_{P \cap G_0}^{G_0}(\chi) \subset C(G_0)$$

dual of $\mathrm{Ind}_{P\cap G_0}^{G_0}(\chi)$ $\uparrow$ $D^c(G_0, K)$

$\searrow$ fin. gen over $D^c(G_0, K)$

$\Rightarrow$ by Fact 2, $\mathrm{Ind}_P^G(\chi)$ is adm

Special case $\boxed{n = 2}$ ... The Theory is well-known

$\chi\left(\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}\right) = \exp(c(\chi) \cdot \log(a))$ for a close to 1

where $c(\chi) \in K$

Thm (S. / Teitelbaum)

If $c(\chi)$ is not non-posi integer, then $\mathrm{Ind}_P^G(\chi)$ is top. irred for G_0. (← inf-dim; cpt)

In general for G, the length is ≤ 2