3.10.

$K/\mathbb{Q}_p$ fin.

v. spaces are always over $K$.

adm smooth rep of $G$ on $V$. ("adm" somehow disappeared b/c irred + sm $\Rightarrow$ adm).

$\forall$ cpt open subgp $U \subseteq G$, the fixed vectors $V^U$ is fin dim.

Banach space repns

$G$ is any p-adic Lie gp. (e.g. $G = GL_n(L)$ or any open subgp therein)

**Def** A $\boxed{\text{norm}}$ on a $K$-v. sp $V$ is a fcn $\| \ \| : V \to \mathbb{R}_{\geq 0}$ s.t.

(i) $\|v + v'\| \leq \max(\|v\|, \|v'\|)$.

(ii) $\|av\| = |a| \cdot \|v\|$

(iii) $\|v\| = 0 \Rightarrow v = 0$

gives a metric and hence a top on $V$.

**Def** A $\boxed{\text{K-Banach space}}$ is a top'l $K$-v.sp. whose top. can be defined by a norm and which is complete.

**Def** A $\boxed{\text{Banach rep}}$ $V$ of $G$ is a lin. action of $G$ on $V$ s.t. the map $G \times V \to V$ is cont.

$\boxed{\mathrm{Ban}_K(G)} :=$ category of these.

This is unreasonably big categ!

e.g. $V_0 \hookrightarrow V_1$ w/ dense image, not surj, $V_0$ and $V_1$ are top. irred (no closed proper $G$-inv. subsp)

Ex (Diarra).

$G = \mathbb{Z}_p$.

Take $z \in \mathbb{C}_p = \widehat{\overline{\mathbb{Q}}_p}$ s.t. $|z| < 1$.

$V_z$ the smallest closed subfield of $\mathbb{C}_p$ containing $z$.

a $\mathbb{Q}_p$-Banach sp.

let $G$ act by $a \cdot v := (1+z)^a \cdot v$.

Fact If $z$ is transcendental/$\mathbb{Q}_p$ then $V_z$ is inf. dim and is a top. irred G-re

Goal: Construct a "reasonable" subcateg. of "admissible" Ban. rep. which is still rich enough.

Rem ① $V$ is a Ban. space w/ norm $\|\cdot\|$.

$L := \{v \in V : \|v\| \le 1\}$ is an $\mathcal{O}_K$-submod s.t.

* $K \otimes_{\mathcal{O}_K} L = V$ ("lattice"). (but $L$ is not nec. free $\mathcal{O}_K$-mod...)
* $L$ is open & bdd in $V$.

Vice versa, let $L \subseteq V$ be a bdd open lattice, then $\|v\|' := \inf_{v \in aL} |a^*|$. is a defining norm for $V$.

② In general, we do not find a defining norm which is preserved by $G$. But we do for any cpt subgp $H \subseteq G$.

[Q] Richard, ($\|\ \|'$ is eqv to $\|\ \|$, and these are discrete valn ?)

☆ Key def A Banach rep $V$ of $G$ is called $\boxed{\text{adm}}$ if $\exists$ a cpt open subgp $H \subseteq G$ and an $H$-inv. bdd open lattice $L \subseteq V$ s.t. the following finiteness condition holds:

$\forall$ open subgp $U \subseteq H$ the $U$-fixed elts $(V/L)^U$ are of cofinite type over $\mathcal{O}_K$

"alg. obj"

Rem

① "cofin type" means $(K/\mathcal{O}_K)^m \oplus$ finite. (Pont dual is f.g.)

② If $V$ is adm, then $\forall$ cpt open subgp $H \subseteq G$, we find..

③ Suppose $V$ is adm. w/ $H$ and $L$ as in the def. reduce to $\bar{L} = L/\pi_K L$ is a smooth $H$-rep over $\mathcal{O}_K/\pi$, which is adm-smooth. (can go backward from $\bar{L}$ to $L$ by Nakayama - Mazur)

$\boxed{\mathrm{Ban}^a_K(G)}$ := categ of all adm ones.

**Thm** (Sch./Teitelbaum)

(i) $Ban_K^a(G)$ is an abel. categ.

(ii) All maps in $Ban_K^a(G)$ are strict w/ closed image.

($0 \to V_1 \to V_2 \to V_3 \to 0 \Rightarrow$ top. on $V_3 \overset{\text{quot}}{=}$ top on $V_2/V_1$)

(Strategy of pf)

May assume that $G$ is cpt.

$C(G) := \{$all cont fcn $G \to K.\}$

← Ban. rep for sup-norm.

$g \mapsto \delta_g$, $G \subseteq D^c(G,K) :=$ cont. dual of $C(G)$. is in fact a $K$-alg s.t.

not dual as Ban. spaces.

$D^c(G,K) \cong K \otimes_{\mathcal{O}_K} \mathcal{O}_K[\![G]\!]$.

gp action transl. ⇒ adm but hard to ~~say~~ det. its str as a mod over a gp ring. (so dual is more useful)

Fact 1 (Lazard) — serious thm.

$\mathcal{O}_K[\![G]\!]$ is noeth.

Fact 2 (main step)

$Ban_K^a(G) \xrightarrow{\sim} Mod_{f.g.}(D^a(G,K))$

$V \longmapsto V' =$ cont. dual of $V$.

is an anti-eqv. of categ.

→ so very algebraic!

Aside

Any f.g. $\mathcal{O}_K[\![G]\!]$-mod has a unique top. making $\mathcal{O}_K[\![G]\!]$-action cont.

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$\boxed{\text{cont. principal series}}$

$G = GL_n(\mathbb{Q}_p)$

$\cup\!|$

$P :=$ lower $\nabla$ matrices.

$\chi : P \to K^\times$ cont. char.

$Ind_P^G(\chi) = \{ f : G \to K \text{ cont} \mid f(gp) = \chi(p)^{-1} f(g),\ \forall g \in G, p \in P \}$

$G$ acts by left transl.

(why Ban. rep?) Iwasawa dec. $G = G_0 P$, $G_0 := GL_n(\mathbb{Z}_p)$

take sup-norm/over $G_0 \rightsquigarrow Ind_P^G(\chi)$ is an adm. Ban. rep why adm?

$$\mathrm{Ind}_P^G(\chi) = \mathrm{Ind}_{P\cap G_0}^{G_0}(\chi) \qquad \text{dual} \qquad \Rightarrow$$ easy to see dual is f.g. ($\because$ cofin) ⑧

$$\cap \qquad\qquad \twoheaduparrow$$

$$C(G_0). \qquad D^c(G_0, K).$$

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$\boxed{n=2}$

$$\chi\left(\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}\right) = \exp.\left(c(\chi)\log(a)\right) \quad \text{for } a \text{ close to } 1.$$

where $c(\chi) \in K$.

Thm (Sch. Teitel) If $c(\chi) \notin -\mathbb{N}_0$, then $\mathrm{Ind}_P^G(\chi)$ is top. irred. for $G_0$. In gen'l, for $G$ the length is $\le 2$.

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3.15. $X/\mathcal{O}_K$ : str sst. $k = \mathcal{O}_K/(\pi)$ ; $\mathcal{O}_K$ DVR ; rel dim $n$. Tem's talk.

$Y = X \otimes_{\mathcal{O}_K} k. = \bigcup_{i=1}^{n+1} Y_i$ ; $Y_i$ proper sm$/k$. irred. dim $n$.

$I \subset \{1, 2, \cdots, n+1\}$.

$Y_I := \bigcap_{i\in I} Y_i$

rel dim $2n / \mathcal{O}_K$ ; $(X \times_{\mathcal{O}_K} X) \otimes k = Y \times_k Y. = \bigcup_{1\le i,j\le n+1} Y_{i,j}$ ; $Y_{i,j}$ proper sm. dim $2n$.

but not regular. not normal crossing

$I, I' \subset \{1, 2, \cdots, n+1\}$.

$Y_{I,I'} := Y_I \times_k Y_{I'} \subset Y \times_k Y$

$$\begin{array}{l} X' : \text{str sst} \\ \downarrow \ \text{Saito's blow-up} \\ X \times_{\mathcal{O}_K} X \end{array}$$

$X' \otimes_{\mathcal{O}_K} k = \bigcup_{1\le i,j\le n+1} D_{i,j}$ — Cart. div. crossing normally w/ each other.

$D_{i,j}$: str. transf. of $Y_{i,j}$ in $X'$

irred compos of $Y' := X' \otimes_{\mathcal{O}_K} k$.