Mar 15. 2006. Wed. Peter Schneider (Lecture 3)

p-adic Banach spaces - Joint with J. Teitelbaum,
with C. Breuil.

$L = \mathbb{Q}_p$. ω_p : additive valuation

$G = \mathbb{Q}_p$-points of a <u>split</u> <u>connected</u> <u>reductive</u> gp$/\mathbb{Q}_p$.

$\cup\!\mathsf{I}$

$P = T \cdot N$ Borel subgp. T: maximal split torus

N : unipotent radical

$W = N(T)/T$ Weyl group. $N(T)$ = normalizer of T in G.

$U_0 :=$ "good" maximal compact subgp

$T_0 := T \cap U_0$

$\Lambda := T/T_0$: free abelian group. $X^*(T) = \mathrm{Hom}(\Lambda, \mathbb{G}_m)$

$$X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T) \xrightarrow{\cong} \Lambda$$
$$\nu \longmapsto \nu(p) \cdot T_0$$

· <u>Classical unramified Langlands functoriality</u>

K an alg. closed field of char 0

Fix $p \geq 2 \in K$

G'/K cnted Langlands dual gp

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T' torus dual to T. $T'(K) = \mathrm{Hom}(\Lambda, K^*)$

$$X^*(T) \cong X_*(T') = \mathrm{Hom}(\mathbb{G}_m, \mathrm{Hom}(\Lambda, \mathbb{Z}) \otimes \mathbb{G}_m)$$
$$\chi \longmapsto [a \longmapsto (\omega_p \cdot \chi) \otimes a]$$

· Satake-Hecke algebra

$\mathcal{H}(G, I_{u_0})$:= all locally constant functions with cpt supp $\psi : u_0 \backslash G / u_0 \longrightarrow K$.

K-algebra w.r.t. $(\psi_1 * \psi_2)(h) = \sum_{g \in G/u_0} \psi_1(g) \cdot \psi_2(g^{-1} h)$

$\operatorname{ind}_{u_0}^{G}(1)$:= locally const. ftns with cpt supp $f : G/u_0 \longrightarrow K$

$G \circlearrowright \operatorname{ind}_{u_0}^{G}(1) \circlearrowleft \mathcal{H}(G, I_{u_0})$

· Satake Isomorphism.

$$S^{\text{norm}} : \mathcal{H}(G, I_{u_0}) \longrightarrow K[\Lambda]$$

$$\psi \longmapsto \sum_{\lambda = \lambda(t)} \delta^{-\frac{1}{2}}(t) \sum_{u \in N/N_0} \psi(t \cdot u)\, \lambda$$

induces an isomorphism of K-algebras

$$\mathcal{H}(G, I_{u_0}) \xrightarrow{\cong} K[\Lambda]^W$$

δ = unimodular character of P

$\delta(t) = \left| \det(\operatorname{ad}(t), \operatorname{Lie} N) \right|_p^{-1} \in p^{\mathbb{Z}} \subseteq \mathbb{Q}^{\times} \subseteq K^{\times}$

$\delta \in T'(K)$

$p^{\frac{1}{2}} \in K \rightsquigarrow \delta^{\frac{1}{2}} \in T'(K)$

Note: One can drop the normalization by $\delta^{-\frac{1}{2}}$. Then we get corresponding isomorphism with W-action changed by a certain cocycle.

Note also: $K[\Lambda] = \mathcal{O}_{alg}(T')$

$K[\Lambda]^W = \mathcal{O}_{alg}(W\backslash T')$

$$\mathrm{Max}(\mathcal{H}(G, I_{u_0})) = (W\backslash T')(K) = \text{Set of semi-simple conjugacy classes in } G'(K)$$

$$\begin{matrix} \cup \\ \xi \end{matrix} \qquad \updownarrow$$

specialization

isomorphism classes of unramified semi-simple Weil gp parameters

$$H_\xi := \mathrm{ind}_{U_0}^G(1) \otimes_{\mathcal{H}(G, I_{u_0})} K_\xi$$

$$\begin{matrix} W_{\mathbb{Q}_p} & \longrightarrow & G'(K) \\ \searrow & & \uparrow \\ & W_{\mathbb{Q}_p}/I_p = \mathbb{Z} & \end{matrix}$$

is a finite length smooth G-rep'n and has a unique irreducible quotient V_ξ. $<$ ...... unramified Langlands Functoriality

Now $K/\mathbb{Q}_p$ finite.

We bring in an irreducible $\mathbb{Q}_p$-rational rep'n (ρ, E) of G corresponding to a highest weight $\xi \in X^*(T)$.

$\mathcal{H}(G, \rho|_{U_0}) :=$ Compactly supported functions $\psi: G \to \mathrm{End}_K(E)$ satisfying $\psi(u_1 g u_2) = \rho(u_1) \cdot \psi(g) \cdot \rho(u_2)$ for $u_1, u_2 \in U_0$, $g \in G$.

$$G \curvearrowright \mathrm{ind}_{U_0}^G(\rho|_{U_0}) \curvearrowleft \mathcal{H}(G, \rho|_{U_0})$$

Fix a $\mathcal{U}_0$-invariant norm $\|\cdot\|$ on E.

$\rightsquigarrow$ operator norm on $\mathrm{End}_K(E)$

$\rightsquigarrow$ Sup-norms on $\mathcal{H}(G, \rho|\mathcal{U}_0)$ and on $\mathrm{ind}_{\mathcal{U}_0}^G(\rho|\mathcal{U}_0)$

$$\begin{array}{ccc} \mathcal{H}(G, \rho|\mathcal{U}_0) & & \mathrm{ind}_{\mathcal{U}_0}^G(\rho|\mathcal{U}_0) \\ \Big\downarrow \text{complete} & & \Big\downarrow \text{complete} \\ \mathcal{B}(G, \rho|\mathcal{U}_0) & & \mathcal{B}_{\mathcal{U}_0}^G(\rho) \end{array}$$

$\mathcal{B}(G, \rho|\mathcal{U}_0)$: K-Banach alg.

$\mathcal{B}_{\mathcal{U}_0}^G(\rho)$: K-Banach space with
- conti. isometric action of G
- conti. action of $\mathcal{B}(G,\rho)$

<u>Note</u>:
$$\mathcal{H}(G, \rho|\mathcal{U}_0) \cong \mathcal{H}(G, 1_{\mathcal{U}_0}) \xrightarrow[\text{Satake}]{\cong} K[\Lambda]^W = \mathcal{O}_{alg}(W\backslash T')$$
$$\psi \cdot \rho \longleftarrow \psi$$

<u>Point</u>: the two norms on $\mathcal{H}(G, \rho|\mathcal{U}_0)$ and $\mathcal{H}(G, 1_{\mathcal{U}_0})$ are very different.

Again $p^{\frac{1}{2}} \in K$.

Define a norm $\|\ \|_\xi$ on $K[\Lambda]$ by

$$\left\| \sum c_\lambda \lambda \right\|_\xi := \sup_{\lambda = \lambda(t)} \left| \delta^{\frac{1}{2}}({}^w t) \cdot p^{\omega_p(\xi({}^w t))} c_\lambda \right|$$

where w (depending on λ) is chosen in such a way that ${}^w\lambda$ is anti-dominant.

$\boxed{\text{Prop 1}}$
$$(\mathcal{H}(G, \rho|\mathcal{U}_0), \text{sup-norm}) \cong (K[\Lambda]^W, \|\cdot\|_\xi).$$

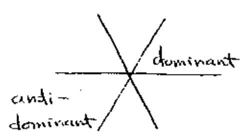
Further notation:

$$\mathrm{val}: T'(K) = \underset{\substack{\| \\ T/T_0}}{\mathrm{Hom}(\Lambda, K^\times)} \xrightarrow{\omega_p \circ} \mathrm{Hom}(\Lambda, \mathbb{R}) =: V_{\mathbb{R}} \quad \text{root space}$$

$$\begin{tikzcd} \xi_{\mathbb{Q}_p} \in V_{\mathbb{R}} & \omega_p \circ \chi \\ \xi \in X^*(T)\otimes\mathbb{R} \arrow[u, mapsto, "\wr"] \arrow[u, shift left=8] & \chi \arrow[u, mapsto] \end{tikzcd}$$

$\eta_{\mathbb{Q}_p} := \frac{1}{2}$ sum of the positive roots

- W acts on $V_{\mathbb{R}}$, there is a partial order $\leq$ on $V_{\mathbb{R}}$.



$$V_{\mathbb{R}} \ni z \longmapsto z^{\mathrm{dom}} := \text{dominant point in } Wz.$$

<u>Define</u>

$$V_{\mathbb{R}}^{\xi,\mathrm{norm}} := \{ z \in V_{\mathbb{R}} : z^{\mathrm{dom}} \leq \eta_{\mathbb{Q}_p} + \xi_{\mathbb{Q}_p} \}$$

$$= \text{convex hull of } {}^{W}(\eta_{\mathbb{Q}_p} + \xi_{\mathbb{Q}_p})$$

$$T'_{\xi,\mathrm{norm}} := \mathrm{val}^{-1}(V_{\mathbb{R}}^{\xi,\mathrm{norm}})$$

$\boxed{\text{Prop 2}}$ i. $T'_{\xi,\mathrm{norm}}$ is an affinoid subdomain in T' and W-invariant.

ii. $B(G, \rho|\mathfrak{u}_0) \cong \mathcal{O}(W \backslash T'_{\xi,\mathrm{norm}})$

Parameter (ξ, ζ) $\quad \xi$: a highest weight

$\zeta \in T'_{\xi,\mathrm{norm}} \subseteq T'$

We can define the "specialization":

$$B_{\xi,\zeta} := K_\zeta \underset{B(G,\rho|\mathfrak{u}_0)}{\hat{\otimes}} \mathcal{B}^G_{\mathfrak{u}_0}(\rho) \text{ is a unitary Banach space rep'n of } G.$$

Big Problem:

Is $B_{\xi,\xi} \neq 0$?