3.17. (13)

parameters. (ξ, ζ).

$\xi \in X^*(T) = X_*(T') \subseteq X_*(G')(K)$.

$\zeta \in T'(K) \subseteq G'(K)$ s.t. $\zeta \in T'_{\xi, \text{norm}}$.

$\downarrow$

$B_{\xi,\zeta}$ unitary Banach space rep of G.

<u>Conj</u> $B_{\xi,\zeta} \neq 0$. ← not expected to be irred.

$T'_{\xi,\text{norm}} := \mathrm{val}^{-1}(V_\xi^{\text{norm}})$.

$V_\xi^{\text{norm}} = \{ z \in V_{\mathbb{R}} : z^{\text{dom}} \leq \eta_{\mathbb{Q}_p} + \xi_{\mathbb{Q}_p} \}$.

$V_\xi := \{ z \in V_{\mathbb{R}} : (z + \eta_{\mathbb{Q}_p})^{\text{dom}} \leq \eta_{\mathbb{Q}_p} + \xi_{\mathbb{Q}_p} \}$.

$FIC_K :=$ categ. of filtered K-isocrystals — no semilinear b/c base is $\mathbb{Q}_p$.

$\underline{D} = (D, \varphi, \mathrm{Fil}^\bullet D)$

D: f.d. K-v.sp; φ: K-lin aut.; $\mathrm{Fil}^\bullet D$: decreasing fil.

$FIC_K^{\text{adm}} \subseteq FIC_K$ full subcat. of (weakly) adm obj's.

Faltings / Totaro : FIC_K^{adm} is a neutral Tannakian categ

$\Uparrow$ implies (Jay says).

Colmez / Fontaine : $FIC_K^{\text{adm}} \simeq \mathrm{Rep}_K^{\text{crys}}(\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p))$

$\boxed{A}$ $G = GL_{d+1}(\mathbb{Q}_p)$

$\cup\!\mid$

P lower ∇ Borel.

$\cup\!\mid$

T. diag. mat.

$U_0 := GL_{d+1}(\mathbb{Z}_p)$.

$T/T_0 = \Lambda \ni \lambda_i := \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & p & \\ & & & \ddots \\ & & & & 1 \end{pmatrix} T_0$. ($p$ at the i^{th} spot.)

$$V_{\mathbb{R}} = \mathrm{Hom}(\cancel{\Lambda}, \mathbb{R}) \xrightarrow{\cong} \mathbb{R}^{d+1}$$
$$z \longmapsto (z(\lambda_1), \cdots, z(\lambda_{d+1})).$$

$$X^*(T) \otimes \mathbb{R} \xrightarrow{\cong} V_{\mathbb{R}} \xrightarrow{\cong} \mathbb{R}^{d+1}.$$

$$\xi = \left[\begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_{d+1} \end{pmatrix} \mapsto \prod g_i^{a_i} \right] \longmapsto (a_1, \cdots, a_{d+1}).$$

dominant $a_1 \le \cdots \le a_{d+1}$.

$$\eta_{\mathbb{Q}_p} \cong \tfrac{1}{2}(-d, -(d-2), \cdots, d-2, d).$$

<u>note</u>: $\widetilde{\eta}_{\mathbb{Q}_p} = (0, 1, \cdots, d) = \eta_{\mathbb{Q}_p} + \underbrace{\tfrac{1}{2}(d, \cdots, d)}_{W\text{-inv.}}$ adding this to def of V_ξ

$$V_\xi = \{ z : (z + \widetilde{\eta}_{\mathbb{Q}_p})^{\mathrm{dom}} \le \widetilde{\eta}_{\mathbb{Q}_p} + \xi_{\mathbb{Q}_p} \}.$$

$\Downarrow$ now the formula involves integ. wts only.

use on T' the coord.

$$T'(K) = \mathrm{Hom}(\Lambda, K^\times) \longrightarrow (K^\times)^{d+1}$$
$$\zeta \longmapsto (\zeta(\lambda_1), p\zeta(\lambda_2), \cdots, p^d \zeta(\lambda_{d+1})).$$

<u>Fact</u> Under these coord. $T'_\xi = \mathrm{val}^{-1}(V_\xi)$ corresp. to

$$(\omega_p(\zeta_1), \cdots, \omega_p(\zeta_{d+1}))^{\mathrm{dom}} \le (a_1, a_2+1, \cdots, a_{d+1}+d)$$

Hodge - Newton polygon!

implies HT wts are strictly increasing, or HT wt mult $= 1$. $\Leftarrow$

<u>moreover</u>: ① $(\cdot)^{\mathrm{dom}}$ means rearranging in an increasing order.

② $(z_1, \cdots, z_{d+1}) \le (z_1', \cdots, z_{d+1}')$ if

$$z_{d+1} \le z_{d+1}', \quad z_d + z_{d+1} \le z_d' + z_{d+1}', \quad \cdots, \quad z_2 + \cdots + z_{d+1} \le z_2' + \cdots + z_{d+1}'.$$
$$z_1 + \cdots + z_{d+1} = z_1' + \cdots + z_{d+1}'.$$

<u>D</u> in FIC_K of dim $\cancel{d}+1$.

(jump index) where filtr. jumps

$\rightsquigarrow$ * $\varphi \in GL_{d+1}(K) = G'(K)$.

* $\mathrm{Fil}^\circ D$ has a type, $\boxed{\mathrm{type}(D)}$,

which could be viewed as a dominant elt in $X^*(T)$.

Thm For $\xi \in X^*(T)$ dominant and $\zeta \in T'(K)$, TFAE.

1. $\zeta \in T'_{\xi}$.
2. $\exists$ a $\underline{D}$ in FIC_K^{adm} s.t. $\varphi^{ss} = \zeta$ and $\mathrm{type}(\underline{D}) = \xi$.

crucial.

(admissibility : condition on all sub isocrys.)

using the Fontaine ftr

our param. $(\xi, \zeta) \mapsto$ family of crys. Gal rep.

(HT wt mult = 1 restriction ...)

$\downarrow$?

$B_{\xi,\zeta} \longmapsto$ family of ~~quots~~ all top. irred quots.

$\boxed{B}$ G general (split)

$REP_K(G') :=$ categ of K-rat'l rep of G'
neutral Tannakian categ.

Let's look at a general pair
$(\nu, b) \in X_*(G')(K) \times G'(K)$

we have ftr

$I_{(\nu,b)} : REP_K(G') \longrightarrow FIC_K$

$(\rho, E) \longmapsto (E, \rho(b), \text{wt fil of cochar } \rho\circ\nu)$.

Def (ν, b) is called adm if $I_{(\nu,b)}$ has image in FIC_K^{adm}.

(ν, b) adm

$\Rightarrow REP_K(G') \xrightarrow{I(\nu,b)} FIC_K^{adm}$

~~faithful~~ $\otimes$-ftr

$\updownarrow \sim$

$Rep_K^{cris}(Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p))$.

(1)

~~(v,b) adm~~

$\Rightarrow$ $REP_K(G') \xrightarrow{\mathcal{I}(v,b)} FIC^{adm}_{K}$.

Tanak. formalism.

$\Longrightarrow$ $\gamma_{(v,b)} : Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \longrightarrow G'(\overline{K})$ — can take K^{ur}. (Steinberg thm. van of coh. fib for isom $/K^{ur}$?)

det'd up to conj. on the target.

harmless b/c this is "Langlands param."

Problems 1). Fontaine forces us to use $T'_{\xi,norm}$.

$\rightsquigarrow$ need $p^{\frac{1}{2}} \in K$.

2) $\eta_{\mathbb{Q}_p}$ not integral.

Thm. Let $\xi \in X_*(T')$. dominant & $\zeta \in T'(K)$. assume $\eta_{\mathbb{Q}_p}$ is integral. TFAE

(i) $\zeta \in T'_{\xi, norm}$.

(ii) $\exists$ an adm pair (v, b) s.t.

$v \in \underbrace{G'(K)}_{\text{conj. action}}$-orbit of $\xi\eta_{\mathbb{Q}_p}$ and $b^{ss} = \zeta$.

What if $\eta_{\mathbb{Q}_p}$ is not integral?

Thm Everything remains the same — requires ($\exists$ of $\sqrt{\ }$ of cyclo char)

* if we work with filtered isocrys. whose fil. are indexed by $\frac{1}{2}\mathbb{Z}$

* and if we replace $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ by its unique non triv central extn (classified by $H^2(G_{\mathbb{Q}_p}, \{\pm 1\}) = \{\pm 1\}$)

but categ is too big?

$$1 \to \{\pm 1\} \to Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)_{(2)} \to Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to 1.$$

provided K is big enough.

(☆)

(✡) means, this is surj, so $\exists$ $\sqrt{\ }$ of ε.

$$\begin{tikzcd} G_{\mathbb{Q}_p(2)} \arrow[r] \arrow[d, dashed, "\varepsilon_{(2)}"'] & G_{\mathbb{Q}_p} \arrow[r] \arrow[d, "\varepsilon"] & 0 \\ K^\times \arrow[r, "2"] & K^\times \arrow[r] & 1. \end{tikzcd}$$

$\sqrt{\ }$ of ε $\hookrightarrow$ $\varepsilon_{(2)}$

ε - cyclo char

$$FIC^{adm.}_{K,2}$$
$$\cup\!\!|$$
$$\left\{ \begin{array}{l} \otimes\text{-subcat. gen'd by} \\ FIC^{adm}_K \text{ and } \sqrt{\varepsilon} \end{array} \right\}$$

twist -1 compo by this.

Define crys. rep by requiring this is crys. $\blacksquare$ $\boxed{\text{END}}$

(pf) Tannakian formalism.

(roughly what I heard)

* To Any Tannak. cat, with a choice of fiber ftr (to the cat. of v. sp.) we can assoc. a gp (pro-alg. gp) whose rep. categ is eqv to the Tan. cat. There may be many such gps.

$G \to G'$ induces $Rep(G) \leftarrow Rep(G')$.

T_1, T_2: Tan. cat.

$$\begin{tikzcd} T_1 \arrow[rr, "f"] \arrow[dr, "f_1"'] & & T_2 \arrow[dl, "f_2"] \\ & Vec_K & \end{tikzcd}$$

fiber ftr.

any f s.t. diag. comm.

$\updownarrow$ 1-1

$Aut(f_1) \leftarrow Aut(f_2)$.

aut. gp of fib ftr.

In our case,

$$\begin{tikzcd} Rep(G') \arrow[r] \arrow[d, "forget"'] \arrow[dr, "f_0"] & Rep(Gal) \arrow[d, "forget"] \\ Vec_K & Vec_K \end{tikzcd}$$

composite

$\Rightarrow$ $Aut(forget)(K^{ur}) \cong Aut(f_0)(K^{ur})$

$\quad \| \qquad\qquad\qquad\qquad \uparrow$ contains Gal.

$G'(K^{ur})$

choice. (up to inner aut).

existence can be seen by Steinberg thm.

$H^1(K^{ur}, G') = 0$