Tilouine's lectures on p-adic Siegel modular forms

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1. Introduction

The study of congruence between elliptic modular forms by Serre, Swinnerton-dyer in the 1970's led Dwork, Katz, and Serre to formulate the notion of p-adic elliptic modular forms. In Katz's formulation, both classical and p-adic modular forms can be described in a geometric way: a classical modular form is a global section of the sheaf of differentials on the classical modular curves, while a p-adic modular form is a global section of the structural sheaf of the tower of Igusa curves; these are Galois coverings of classical modular curves having covering group of the form $\mathbf{Z}_p^*$. For the p-adic theory, one needs a detailed study of the integral structure of modular curves and Igusa curves, which is supplied by the work of Igusa, Deligne-Rapoport, Katz-Mazur.

The p-adic modular forms which are of most interest are the eigenforms; these are forms which are eigenfunctions of all the Hecke operators. Beginning with the work of Hida in the 80's, and then with the work of Coleman, Coleman-Mazur, it was realised that p-adic eigenforms should vary in families.

More precisely, let f be a p-adic modular eigenform. Then as in the classical case, it has a q-expansion:

$$f = \sum_n a_n q^n, \ \ a_n \in \bar{\mathbf{Z}}_p$$

Let val_p be the normalised valuation of $\bar{\mathbf{Q}}_p$, i.e. $\mathrm{val}_p(p) = 1$. Then the slope of f is defined to be $\mathrm{val}_p(a_p)$. f is said to be p-ordinary if its slope is zero.

Works of Hida show that every p-ordinary eigenform can be fit into a p-adic family of such eigenform, with the members being parametrised by the weights of the form. In a series of papers, Coleman extended this theory to the case where the slope of the eigenform is finite (i.e. $a_p \neq 0$). He showed that it is always possible to vary p-adic eigenform of finite slope in local families. Finally, Coleman-Mazur patched these local families of Coleman to form a global parametrisation space known as the eigencurve.

Besides, Hida's theory has important applications as well. For example, it plays an important role in Wiles' work on the Iwasawa main conjecture over totally real fields, and on the Shimura-Taniyama conjecture.

Due to its importance, it would be desirable to generalise the theory of Hida (and also Coleman-Mazur) to other types of modular forms.

Siegel modular form is a generalisation of the classical elliptic modular form, where the role of elliptic curves is replaced by abelian varieties. The moduli space of abelian varieties is known as Siegel modular variety, and Siegel modular forms arise as section of sheaf of differentials on the Siegel modular variety.

By analogy with the case of elliptic modular form, one need to have a theory of p-adic Siegel modular form, and this requires the study of the integral structures of Igusa varieties, which are coverings of the Siegel modular varieties [**?**],[**?**].

There are partial results on extension of Hida's theory of p-ordinary eigenform to the Siegel modular case [**?**]. The case where important results are obtained is the case of rank 2 Siegel modular eigenform, where the Siegel modular variety is the moduli of abelian varieties of dimension 2; in the language of automorphic representation, these realise automorphic representation of the group of symplectic similitudes GSp(4).

On the other hand, work remains to be done to extend the theory of Coleman-Mazur to the Siegel case.

References

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