Hida's lecture on $\mathcal{L}$-invariant and Galois deformation theory

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1. Introduction

Let E be a (modular) elliptic curve over $\mathbb{Q}$, split-multiplicative at an odd prime p, so that one has Tate's analytic parametrisation:

$$E(\overline{\mathbb{Q}}_p) = \overline{\mathbb{Q}}_p^\times / q_E^{\mathbb{Z}} \ , \ q_E \in \mathbb{Q}_p^\times$$

The $\mathcal{L}$-invariant of E at p, $\mathcal{L}_p(E)$, is defined to be:

$$\mathcal{L}_p(E) = \frac{\log_p q_E}{\mathrm{ord}_p\, q_E}$$

The definition of the $\mathcal{L}$-invariant is first proposed in connection with the p-adic Birch-Swinnerton-dyer conjecture, which relates the order of vanishing of the p-adic L-function of E at $s=1$ to the Mordell-Weil rank of E. Recall that [**5**] $L_p(E,s)$ is constructed by p-adically interpolating the twisted special L-value $L_\infty(E,\chi,1)/\Omega_E$, where χ is a finite order character of $\mathbb{Z}_p^\times$ and Ω_E a real period of E. One has the formula:

$$L_p(E,1) = (1 - \frac{1}{a_p}) \frac{L_\infty(E,1)}{\Omega_E}$$

In the case where E is split multiplicative, we have $a_p = 1$, so we have $L_p(E,1) = 0$. Based on numerical data, Mazur-Tate-Teitelbaum conjectured the relation:

$$L_p'(E,1) = \mathcal{L}_p(E) \frac{L_\infty(E,1)}{\Omega_E}$$

This conjecture is proved by Greenberg-Stevens [**2**]. In this proof, an important role is played by Hida's theory of ordinary deformation.

For simplicity, assume that $\mathcal{F} = \sum A_n q^n, A_n \in \Lambda \cong \mathbb{Z}_p[[X]]$, is a Hida family of eigenform, with $\mathcal{F}|_{X=0} = f_E$. Then one of the key ingredient in the proof of Greenberg-Stevens is the following formula:

$$\frac{1}{A_p}\frac{d}{dX}A_p|_{X=0} = -\frac{1}{2}\mathcal{L}_p(E) = -\frac{1}{2}\frac{\log_p q_E}{\mathrm{ord}_p\, q_E} \tag{1.1}$$

i.e. the $\mathcal{L}$-invariant gives information about first order deformation.

On the other hand, one can construct, in the spirit of Iwasawa theory, an "arithmetic p-adic L-function" from the data of the Galois representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}(T_pE)$, as follows: let $\mathbb{Q}_\infty$ be the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. The Selmer group $\mathrm{Sel}_{\mathbb{Q}_\infty}(\rho \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ is a Λ-module whose Pontryagin dual is finitely generated torsion over Λ (thanks to the work of Kato). One can then define the characteristic power series associated to the dual Selmer group. The main conjecture predicts equality with the p-adic L-function up to $\Lambda^\times$.

From this optic, it's desirable to have a definition of the $\mathcal{L}$-invariant, in terms of the Galois representation. Greenberg [**1**] gave such a definition, which work more generally for p-ordinary Galois representation with a subquotient having trivial $G_{\mathbb{Q}_p}$ action. For example, with E split multiplicative at p as above, one consider the adjoint square $\mathrm{ad}\,\rho$, consisting of trace zero matrix with Galois acting by conjugation. From Greenberg's definition, one can show:

$$\mathcal{L}_p(E) = \mathcal{L}_p(\mathrm{ad}\,\rho)$$

The case of adjoint square is especially interesting in connection with deformation of Galois representation: let $\mathcal{R}_{\overline{\rho}}/\Lambda$ be the universal deformation ring for the reduction $\overline{\rho}$. One has the relation the dual Selmer group of ρ and the module of Kahler differentials of $\mathcal{R}_{\overline{\rho}}/\Lambda$:

$$\mathrm{Sel}_{\mathbb{Q}_\infty}(\rho \otimes \mathbb{Q}_p/\mathbb{Z}_p) \cong \Omega^1_{\mathcal{R}_{\overline{\rho}}/\Lambda} \otimes_{\mathcal{R}_{\overline{\rho}}} \mathbb{Z}_p \tag{1.2}$$

where $\mathcal{R}_{\overline{\rho}} \to \mathbb{Z}_p$ is the map induced by ρ.

On the other hand, one has the Taylor-Wiles isomorphism:

$$\mathcal{R}_{\overline{\rho}} \cong \mathbb{T}_{\overline{\rho}} \tag{1.3}$$

where $\mathbb{T}_{\overline{\rho}}$ is the local component of Hida's ordinary p-adic Hecke algebra through which $\overline{\rho}$ factors. Exploiting (1.2) and (1.3), Hida showed [**3**]:

$$\frac{1}{A_p}\frac{d}{dX}A_p|_{X=0} = -\frac{1}{2}\mathcal{L}_p(\mathrm{ad}\,\rho) \tag{1.4}$$

which, incidentally, gives another proof of (1.1).

This argument, based on $\mathcal{R} \cong \mathbb{T}$ theorem, has the advantage of being generalisable to the case of totally real fields, as was shown by Hida (based on the work of Fujiwara) [**4**]. In this vein, it's conceivable that the automorphy lifting theorems of Clozel-Harris-Taylor (at least in the minimal case) can be applied to prove similar results.

References

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[4] H.Hida, *$\mathcal{L}$-invariant and Galois deformation*, course notes.
[5] B.Mazur, J.Tate, J.Teitelbaum, *On the p-adic analogues of the conjecture of Birch-Swinnerton-dyer*, Invent Math. **84**, 1986, 1-48.