QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday September 21, 2004 (Day 1)

Each of the six questions is worth 10 points.

1) Let H be a (real or complex) Hilbert space. We say that a set of vectors $\{\phi_n\} \subset H$, $n = 1, 2, \ldots$, has "property D" provided H is the closure of the space of all finite linear combinations of the ϕ_n. Now let $\{\phi_n\}$, $n = 1, 2, \ldots$, be an orthonormal set having property D, and $\{\psi_n\}$ a set of vectors satisfying

$$\sum_{n=1}^{\infty} \|\phi_n - \psi_n\|^2 \ < \ 1\,,$$

where $\|\ \ \|$ refers to the Hilbert space norm. Show that $\{\psi_n\}$ also has property D.

2) Let K be the splitting field of the polynomial $x^4 - x^2 - 1$. Show that the Galois group of K over $\mathbb{Q}$ is isomorphic to the dihedral group D_8, and compute the lattice of subfields of K.

3) Let S be a smooth surface in $\mathbb{R}^3$ defined by $\mathbf{r}(u, v)$, where $\mathbf{r}$ is the radius vector of $\mathbb{R}^3$ and (u, v) are curvilinear coordinates on S. Let H and K be respectively the mean curvature and the Gaussian curvature of S. Let A and B be respectively the supremum of the absolute value of H and K on S. Let a be a positive number and $\mathbf{n}$ be the unit normal vector of S. Consider the surface $\tilde{S}$ defined by $\vec{\rho}(u, v) = \mathbf{r}(u, v) + a\,\mathbf{n}(u, v)$. Let C be a curve in S defined by $u = u(t)$ and $v = v(t)$. Let $\tilde{C}$ be the curve in $\tilde{S}$ defined by $t \mapsto \vec{\rho}(u(t), v(t))$. Show that the length of $\tilde{C}$ is no less than the length of C multiplied by $1 - a\left(A + \sqrt{A^2 + 4B}\right)$. (*Hint:* compare the first fundamental form of $\tilde{S}$ with the difference of the first fundamental form of S and $2a$ times the second fundamental form of S.)

4) Compute the integral

$$\int_0^{\infty} \frac{x^{a-1}}{1 + x^4}\, dx \qquad \left(0 < a < 4\right).$$

5) The *Grassmann manifold* $\mathbf{G}(2,4)$ is the set of all 2-dimensional planes in $\mathbb{R}^4$. More precisely,

$$\mathbf{G}(2,4) = \left\{ M = \begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix} \;\middle|\; a, \dots, h \in \mathbb{R}\,,\ M \text{ has rank } 2 \right\} / \sim$$

where $M_1 \sim M_2$ if and only if $M_1 = AM_2$ for some invertible 2×2 real matrix A. We equip $\mathbf{G}(2,4)$ with the quotient topology; it is a compact orientable manifold.

a) Compute $\pi_1(\mathbf{G}(2,4))$ (*You may want to use the fact that given any* 2×4 *real matrix* M *there exists an invertible* 2×2 *real matrix* A *such that* AM *is in reduced row-echelon form. This gives a cell decomposition of* $\mathbf{G}(2,4)$ *with one cell for each possible reduced row-echelon form.*)

b) Compute the homology and cohomology groups of $\mathbf{G}(2,4)$ (with integer coefficients), stating carefully any theorems that you use.

6) a) What is the dimension of the space of hyperplanes in $\mathbf{P}^{n+1}$ containing a fixed linear subspace L of dimension k?

b) Let $Q \subset \mathbf{P}^{n+1}$ be a smooth quadric over $\mathbb{C}$. Show that the map $Q \to (\mathbf{P}^{n+1})^*$ associating to a point $x \in Q$ the tangent hyperplane $T_xQ \subset \mathbf{P}^{n+1}$ is an isomorphism onto its image. (Here $(\mathbf{P}^{n+1})^*$ is the dual projective space parameterizing hyperplanes in $\mathbf{P}^{n+1}$.)

c) Show that if L is a linear subspace contained in a quadric Q as in b), then $\dim L \leq n/2$.

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

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Wednesday September 22, 2004 (Day 2)

Each of the six questions is worth 10 points.

1) Show that if $Y \subset \mathbf{P}^n$ is a projective subvariety of dimension at least 1, over an algebraically closed field, and $H \subset \mathbf{P}^n$ is a hypersurface, then $Y \cap H \neq \emptyset$. (Justify any intermediate statement you may use.)

2) Let $u \mapsto \vec{\rho}(u)$ be a smooth curve in $\mathbb{R}^3$. Let S be the surface in $\mathbb{R}^3$ defined by $(u, v) \mapsto \vec{\rho}(u) + v\,\vec{\rho}\,'(u)$, where $\vec{\rho}\,'(u)$ means the first-order derivative of $\vec{\rho}(u)$ with respect to u. Assume that the two vectors $\vec{\rho}\,'(u)$ and $\vec{\rho}\,'(u) + v\vec{\rho}\,''(u)$ are $\mathbb{R}$-linear independent at the point $(u, v) = (u_0, v_0)$, where $\vec{\rho}\,''(u)$ means the second-order derivative of $\vec{\rho}(u)$ with respect to u. Verify directly from the definition of Gaussian curvature that the Gaussian curvature of S is zero at the point $(u, v) = (u_0, v_0)$.

3) a) Show that every linear fractional transformation which maps the upper half plane onto itself is of the form

$$F(z) \;=\; \frac{a\,z + b}{c\,z + d}\,, \quad \text{with} \;\; a,\, b,\, c,\, d \in \mathbb{R}\,, \quad a\,d - b\,c \;=\; 1\,.$$

b) Show that every linear fractional transformation which maps the unit disk onto itself is of the form

$$F(z) \;=\; \frac{\alpha\, z + \beta}{\bar{\beta}\, z + \bar{\alpha}}\,, \quad \text{with} \;\; \alpha,\, \beta \in \mathbb{C}\,, \quad |\alpha|^2 - |\beta|^2 \;=\; 1\,.$$

4) a) State van Kampen's Theorem. Use it to exhibit a topological space X such that $\pi_1(X)$ is isomorphic to the free group on 2 generators.

b) Show that the free group on 2 generators contains the free group on n generators as a subgroup of finite index.

c) Show that every subgroup of a free group is free.

5) Let $v_1, \dots v_n$ be complex numbers, and let A be the matrix:

$$A \;=\; \begin{pmatrix} v_1 & v_2 & v_3 & \dots & v_n \\ v_n & v_1 & v_2 & \dots & v_{n-1} \\ v_{n-1} & v_n & v_1 & \dots & v_{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ v_2 & v_3 & v_4 & \dots & v_1 \end{pmatrix} .$$

Compute the eigenvalues and eigenvectors of A.

6) Given $f \in C_0^\infty(\mathbb{R})$ and $\epsilon > 0$, consider the function

$$g_\epsilon(x) \;:=\; \int_{|x-y|>\epsilon} \frac{f(y)}{x-y}\, dy \;.$$

a) Show that

$$Hf(x) \;:=\; \lim_{\epsilon\to 0}\, g_\epsilon(x)$$

exists for each $x \in \mathbb{R}$, and that $Hf \in C^\infty(\mathbb{R})$.

b) Exhibit a universal constant C such that

$$\|Hf\|_{L^2} \;=\; C\,\|f\|_{L^2}\,.$$

Show how to extend the operator H to an isomorphism from $L^2(\mathbb{R})$ to itself.

QUALIFYING EXAMINATION

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Department of Mathematics

Tuesday September 23, 2004 (Day 3)

Each of the six questions is worth 10 points.

1) a) Let G be a group of order n, acting on a finite set S. Show that the number of orbits of this action equals

$$\frac{1}{n}\sum\nolimits_{g\in G} \#\{x \in S \mid gx = x\}.$$

b) Let S be the set of integer points in the rectangle $[0,3] \times [0,2]$. We consider two subsets of S equivalent if one can be transformed into the other by a series of reflections around the horizontal and vertical axes of symmetry of the rectangle. How many equivalence classes of four-element subsets of S are there?

2) Let $U \subset \mathbb{C}$ be a connected open subset. Carefully define the topology of locally uniform convergence on $\mathcal{O}(U)$, the space of holomorphic functions on U. Show that $\mathcal{O}(U)$, equipped with this topology, is a Fréchet space.

3) Consider the two dimensional torus $\mathbf{T}^2 = S^1 \times S^1$, where $S^1 = \mathbb{R}/2\pi\,\mathbb{Z}$. For any fixed $\alpha \in \mathbb{R}$, find all functions $f \in L^2(\mathbf{T}^2)$ with the property

$$f(x_1 + \alpha, x_1 + x_2) \;=\; f(x_1, x_2)\,.$$

4) Let Γ be a set of seven points in $\mathbb{CP}^3$, no four of them lying in a plane. What is the dimension of the subspace of homogeneous quadratic polynomials in $\mathbb{C}[X_0, X_1, X_2, X_3]$ vanishing along any subset $\{p_1, \ldots, p_m\} \subset \Gamma$, $m \leq 7$?

5) (Smooth Version of Michael Artin's Generalization of the Implicit Function Theorem.) Let a and b be positive numbers. Let $\mathcal{R}$ be the ring of all $\mathbb{R}$-valued infinitely differentiable functions on the open interval $(-a, a)$. For elements F, G, H in $\mathcal{R}$ we say that F is congruent to G modulo H in $\mathcal{R}$ if there exists some element Q of $\mathcal{R}$ such that $F - G = QH$ as functions on $(-a, a)$. Let $f(x, y)$ be an $\mathbb{R}$-valued infinitely differentiable function on $\{|x| < a,\ |y| < b\}$ with $f(0, 0) = 0$. Denote by

$f_y(x, y)$ the first-order partial derivative of $f(x, y)$ with respect to y. Let $g(x)$ be an element of $\mathcal{R}$ such that $g(0) = 0$ and $\sup_{|x|<a} |g(x)| < b$. Assume that $f(x, g(x))$ is congruent to 0 modulo $(f_y(x, g(x)))^2$ in $\mathcal{R}$. Prove that there exists an $\mathbb{R}$-valued infinitely differentiable function $q(x)$ on $|x| < \eta$ for some positive number $\eta \leq a$ such that

(i) $q(0) = 0$,

(ii) $f(x, q(x)) \equiv 0$ on $|x| < \eta$,

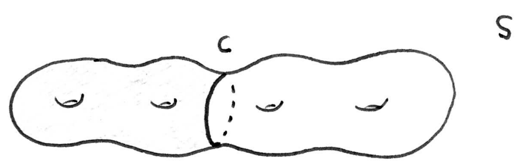
(iii) $q(x)$ is congruent to $g(x)$ modulo $f_y(x, g(x))$ in the ring of all $\mathbb{R}$-valued infinitely differentiable functions on the open interval $(-\eta, \eta)$.

(Note that the usual implicit function theorem is the special case where $g(x) \equiv 0$ and $f_y(x, g(x))$ is nowhere zero on $(-a, a)$ and is therefore a unit in the ring $\mathcal{R}$.)
Hint: Let $q(x) = g(x) + f_y(x, g(x))h(x)$ and solve for $h(x)$ by the usual implicit function theorem after using an appropriate Taylor expansion of the equation.

6) For each of the following, either exhibit an example or show that no such example exists:

a) A space X and a covering map $f : \mathbb{CP}^2 \to X$.

b) A retract from the surface S to the curve C



c) A retract from the surface S' onto the curve C'