QUALIFYING EXAMINATION

Harvard University
Department of Mathematics
Tuesday, October 24, 1995 (Day 1)

1. Let K be a field of characteristic 0.

a. Find three nonconstant polynomials $x(t), y(t), z(t) \in K[t]$ such that

$$x^2 + y^2 = z^2$$

b. Now let n be any integer, $n \geq 3$. Show that there do not exist three nonconstant polynomials $x(t), y(t), z(t) \in K[t]$ such that

$$x^n + y^n = z^n.$$

2. For any integers k and n with $1 \leq k \leq n$, let

$$S^n \;=\; \{(x_1, \ldots, x_{n+1}) \;:\; x_1^2 + \ldots + x_{n+1}^2 = 1\} \;\subset\; \mathbb{R}^{n+1}$$

be the n-sphere, and let $D_k \subset \mathbb{R}^{n+1}$ be the closed disc

$$D_k \;=\; \{(x_1, \ldots, x_{n+1}) \;:\; x_1^2 + \ldots + x_k^2 \leq 1;\; x_{k+1} = \ldots = x_{n+1} = 0\} \;\subset\; \mathbb{R}^{n+1}.$$

Let $X_{k,n} = S^n \cup D_k$ be their union. Calculate the cohomology ring $H^*(X_{k,n}, \mathbb{Z})$.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be any $\mathcal{C}^\infty$ map such that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \;\equiv\; 0.$$

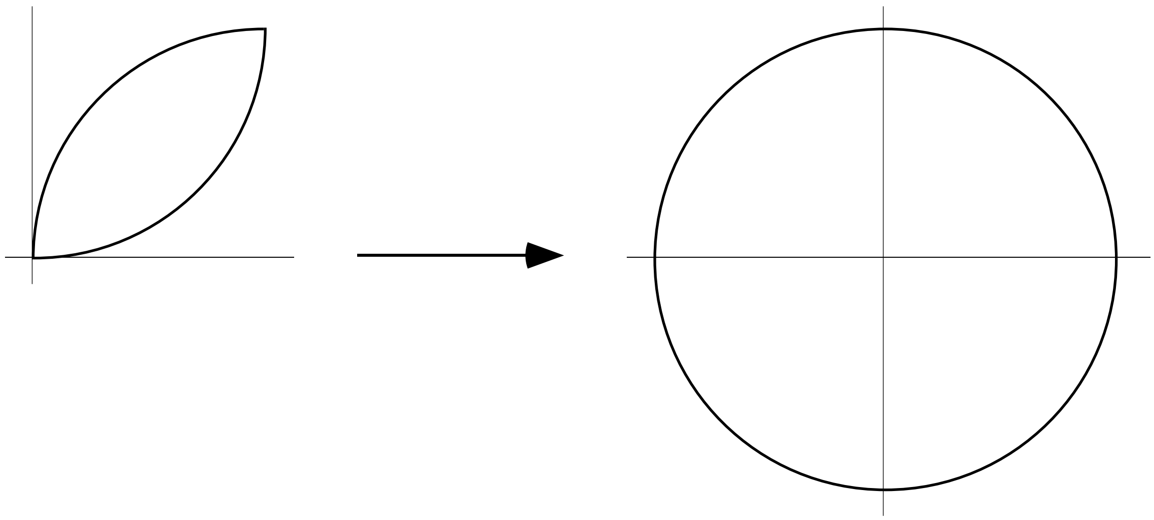
Show that if f is not surjective then it is constant.

4. Let G be a finite group, and let $\sigma, \tau \in G$ be two elements selected at random from G (with the uniform distribution). In terms of the order of G and the number of conjugacy classes of G, what is the probability that σ and τ commute? What is the probability if G is the symmetric group S_5 on 5 letters?

5. Let $\Omega \subset \mathbb{C}$ be the region given by

$$\Omega \;=\; \{z \;:\; |z-1| < 1 \quad \text{and} \quad |z-i| < 1\}.$$

Find a conformal map $f : \Omega \to \Delta$ of Ω onto the unit disc $\Delta = \{z : |z| < 1\}$.



6. Find the degree and the Galois group of the splitting fields over $\mathbb{Q}$ of the following polynomials:

a. $x^6 - 2$

b. $x^6 + 3$

QUALIFYING EXAMINATION
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Wednesday, October 25, 1995 (Day 2)

1. Find the ring A of integers in the real quadratic number field $K = \mathbb{Q}(\sqrt{5})$. What is the structure of the group of units in A? For which prime numbers $p \in \mathbb{Z}$ is the ideal $pA \subset A$ prime?

2. Let $U \subset \mathbb{R}^2$ be an open set.

a. Define a *Riemannian metric* on U.

b. In terms of your definition, define the *distance* between two points $p, q \in U$.

c. Let $\Delta = \{(x, y) : x^2 + y^2 < 1\}$ be the open unit disc in $\mathbb{R}^2$, and consider the metric on Δ given by

$$ds^2 = \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}.$$

Show that Δ is complete with respect to this metric.

3. Let K be a field of characteristic 0. Let $\mathbb{P}^N$ be the projective space of homogeneous polynomials $F(X, Y, Z)$ of degree d modulo scalars ($N = d(d+3)/2$). Let U be the subset of $\mathbb{P}^N$ of polynomials F whose zero loci are smooth plane curves $C \subset \mathbb{P}^2$ of degree d, and let $V \subset \mathbb{P}^N$ be the complement of U in $\mathbb{P}^N$.

a. Show that V is a closed subvariety of $\mathbb{P}^N$.

b. Show that $V \subset \mathbb{P}^N$ is a hypersurface.

c. Find the degree of V in case $d = 2$.

d. Find the degree of V for general d.

4. Let $\mathbb{P}^n_{\mathbb{R}}$ be real projective n-space.

a. Calculate the cohomology ring $H^*(\mathbb{P}^n_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z})$.

b. Show that for $m > n$ there does not exist an *antipodal* map $f : S^m \to S^n$, that is, a continuous map carrying antipodal points to antipodal points.

5. Let V be any continuous nonnegative function on $\mathbb{R}$, and let $H : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be defined by

$$H(f) = \frac{-1}{2}\frac{d^2 f}{dx^2} + V \cdot f.$$

a. Show that the eigenvalues of H are all nonnegative.

b. Suppose now that $V(x) = \frac{1}{2}x^2$ and f is an eigenfunction for H. Show that the *Fourier transform*

$$\hat{f}(y) = \int_{-\infty}^{\infty} e^{-ixy} f(x)dx$$

is also an eigenfunction for H.

6. Find the Laurent expansion of the function

$$f(z) = \frac{1}{z(z+1)}$$

valid in the annulus $1 < |z-1| < 2$.

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Thursday, October 26, 1995 (Day 3)

1. Evaluate the integral

$$\int_0^\infty \frac{\sin x}{x} dx.$$

2. Let p be an odd prime, and let V be a vector space of dimension n over the field $\mathbb{F}_p$ with p elements.

a. Give the definition of a *nondegenerate quadratic form* $Q : V \to \mathbb{F}_p$

b. Show that for any such form Q there is an $\epsilon \in \mathbb{F}_p$ and a linear isomorphism

$$\begin{aligned} \phi\ : V &\longrightarrow \mathbb{F}_p^n \\ v &\longmapsto (x_1, \ldots, x_n) \end{aligned}$$

such that Q is given by the formula

$$Q(x_1, x_2, \ldots, x_n) \;=\; x_1^2 + x_2^2 + \ldots + x_{n-1}^2 + \epsilon x_n^2$$

c. In what sense is ϵ determined by Q?

3. Let G be a finite group. Define the *group ring* $R = \mathbb{C}[G]$ of G. What is the center of R? How does this relate to the number of irreducible representations of G? Explain.

4. Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be any isometry, that is, a map such that the euclidean distance between any two points $x, y \in \mathbb{R}^n$ is equal to the distance between their images $\phi(x), \phi(y)$. Show that ϕ is *affine linear*, that is, there exists a vector $b \in \mathbb{R}^n$ and an orthogonal matrix $A \in O(n)$ such that for all $x \in \mathbb{R}^n$,

$$\phi(x) \;=\; Ax + b.$$

5. Let G be a finite group, $H \subset G$ a proper subgroup. Show that the union of the conjugates of H in G is not all of G, that is,

$$G \neq \bigcup_{g \in G} gHg^{-1}.$$

Give a counterexample to this assertion with G a compact Lie group.

6. Show that the sphere S^{2n} is not the underlying topological space of any Lie group.