QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday 4 February 2003 (Day 1)

There are six problems. Each question is worth 10 points, and parts of questions are of equal weight.

1a. Let k be any field. Show that the ring $k[x]$ has infinitely many maximal ideals.

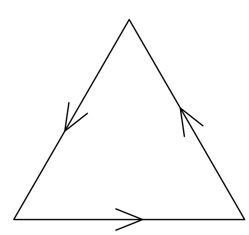
2a. Let f be an entire function. Suppose that f vanishes to even order at every zero of f. Prove there exists a holomorphic function g such that $g^2 = f$.

3a. Let k be a field. Let a, b be relatively prime positive integers. Is there an element in the field of fractions of the k-algebra $A = k[X,Y]/(Y^a - X^b)$ that generates the integral closure of A (i.e., generates it as k-algebra)? If so, find such an element; if not, prove not.

4a. Let $(f(v)\cos(u), f(v)\sin(u), g(v))$ be a parametrization of a surface of revolution $S \subset \mathbb{R}^3$ where $(u,v) \in (0, 2\pi) \times (a,b)$. If S is given the induced metric from $\mathbb{R}^3$, prove that the following map from S to $\mathbb{R}^2$ is locally conformal where $\mathbb{R}^2$ is given the standard Euclidean metric:

$$(u,v) \to \left(u, \int \frac{\sqrt{(f'(v))^2 + (g'(v))^2}}{f(v)} dv \right).$$

5a. Let X be the space obtained by identifying the three edges of a triangle using the same orientation on each edge, as shown below.



Compute $\pi_1(X)$, $H_*(X)$, and $H_*(X \times X)$.

6a. New Real Analysis Problem 1.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Wednesday 5 February 2003 (Day 2)

There are six problems. Each question is worth 10 points, and parts of questions are of equal weight.

1b. Let $X \subset \mathbb{C}^2$ be the curve defined by $x^2(y^2 - 1) = 1$, and let $\overline{X} \subset \mathbb{P}^2$ be its closure.

(i) Find the singularities of $\overline{X}$ and classify them into nodes, cusps, and so on.

(ii) Find the genus of the smooth completion of X.

2b. Let $0 < s < 1$. Evaluate the integral

$$\int_0^\infty \frac{x^{s-1}}{1+x}\, dx.$$

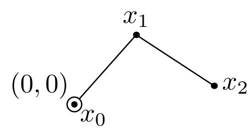
3b. (i) Consider $\mathbb{R}^n$ with the standard Euclidean metric and let $p \in \mathbb{R}^n$ be an arbitrary point. For any $x \in \mathbb{R}^n$ let $\rho_p(x)$ be the distance from p to x. Viewing $\rho_p(x)$ as a smooth function of x away from p, verify that $|\,\mathrm{grad}(\rho_p(x))|^2 = 1$ and that the integral curves of $\mathrm{grad}(\rho_p(x))$ are straight lines. (Here $\mathrm{grad}(\rho_p(x))$ refers to the usual gradient vector field of the function $\rho_p(x)$.)

(ii) More generally, given a smooth function f on a Riemannian manifold (M, g_{ij}), define $\mathrm{grad}(f)$ to be the vector field given locally by

$$\sum_{i,j} \left(g^{ij} \frac{df}{dx_i} \right) \frac{\partial}{\partial x_j}.$$

Show that if $|\,\mathrm{grad}(f)|^2 = 1$ then the integral curves of the vector field $\mathrm{grad}(f)$ are geodesics.

4b. A *mechanical linkage* is a collection of points (some fixed, some not) in the plane connected by rigid struts, each with a fixed length. Its *configuration space* is the set of all solutions to the constraints that the struts have a fixed length, with the topology induced from the product of the plane with itself. For instance, this mechanical linkage

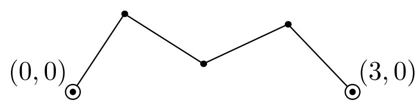


(in which ⊙ denotes a fixed vertex) can be described by the equations

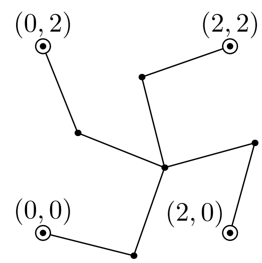
$$\{x_0, x_1, x_2 \in \mathbb{R}^2 | x_0 = (0,0), |x_0 - x_1| = 1, |x_2 - x_1| = 1\}$$

The configuration space of this linkage is the torus $S^1 \times S^1$.

Identify topologically the configuration space of the linkages



and



All edges have length 1, and the fixed vertices are at the indicated locations.

Hint: Consider the position of the central point, and compute the Euler characteristic.

5b. Let H_d be the space of degree d curves in $\mathbb{P}^2$, where $d > 1$. We identify H_d with the projectivization of the vector space of degree d homogeneous polynomials in three variables, so $H_d = \mathbb{P}^N$ for some N.

(i) Find N, the dimension of H_d.

(ii) For a fixed point $p \in \mathbb{P}^2$ find the dimension of the set $\Sigma_p \subset H_d$ of curves that have a singularity at p.

(iii) Find the dimension of the set $\Sigma \subset H_d$ of singular curves.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Thursday 6 February 2003 (Day 3)

There are six problems. Each question is worth 10 points, and parts of questions are of equal weight.

1c. Let $Z \subset \mathbb{P}^n$ be a variety of degree d. Choose a point $P \notin Z$ and let PZ be the union of lines containing the points P and Q, where the union is taken over all points $Q \in Z$. Prove that the degree of PZ is at most d. (Hint: Intersect with a suitable hyperplane and use induction on dimension.)

2c. Let p, q, and r be non-constant non-vanishing entire holomorphic functions that satisfy the equation

$$p + q + r = 0.$$

Prove there exists an entire function h such that p, q and r are constant multiples of h.

3c. Let M be a smooth manifold with a connection ∇ on the tangent bundle. Recall the following definitions of the torsion tensor T and curvature tensor R: For arbitrary vector fields X, Y and Z on M we have

$$T(X,Y) := \nabla_Y X - \nabla_X Y - [X,Y]$$

and

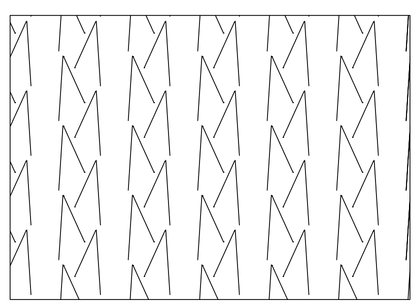
$$R(X,Y)Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[Y,X]} Z.$$

Assuming we have a torsion-free connection ($T = 0$), verify the following identity:

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$$

(Hint: Begin by assuming that X, Y, Z are coordinate vector fields, then justify that there is no loss in generality in doing this.)

4c. (i) What is the symmetry group G of the following pattern? What is the topological space $\mathbb{R}^2$ modulo G?



(ii) What is the commutator subgroup of G? Draw generators for the commutator subgroup on a copy of the pattern (see Page 6).

5c. Let ρ be a two-dimensional (complex) representation of a finite group G such that $\rho(g)$ has 1 as an eigenvalue for every $g \in G$. Prove that ρ is the sum of two one-dimensional representations.

6c. Let k be a field. Let f, g be polynomials in $k[x, y]$ with no common factor. Show that the quotient ring $k[x, y]/(f, g)$ is a finite dimensional vector space over k.