Questions in AG for the qualifying exam, Spring 2013 (draft version by Suh).

P1. Prove that the following complex algebraic varieties are pairwise nonisomorphic.

(a) $X_1 = \operatorname{Spec}\mathbb{C}[x,y]/(y^2 - x^3)$, $X_2 = \operatorname{Spec}\mathbb{C}[x,y]/(y^2 - x^3 - x)$ and $X_3 = \operatorname{Spec}\mathbb{C}[x,y]/(y^2 - x^3 - x^2)$.

(b) $X_1 = \operatorname{Spec}\mathbb{C}[x,y]/(xy^2 + x^2y)$ and $X_2 = \operatorname{Spec}\mathbb{C}[x,y,z]/(xy,yz,zx)$.

(c) $X_1 = \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, $X_2 = \mathbb{P}^2_{\mathbb{C}}$ and $X_3 =$ the blowing up of X_2 at the point $[0:0:1]$.

P2. Let f and g be irreducible homogeneous polynomials in $S = \mathbb{C}[X_0, X_1, X_2, X_3]$ of degrees 2 and 3, respectively. For parts (a) and (b), combinatorial polynomials (such as $\binom{T}{2} = T(T-1)/2$) are acceptable in the final answer.

(a) Compute the Hilbert polynomial of $X = \operatorname{Proj}(S/(g))$ embedded in $P = \mathbb{P}^3_{\mathbb{C}} = \operatorname{Proj}(S)$.

(b) Compute the Hilbert polynomial of $Y = \operatorname{Proj}(S/(f,g))$ embedded in P.

(c) Assuming in addition that Y is nonsingular, use your answer for part (b) to compute its geometric genus

$$\dim_{\mathbb{C}} \Gamma(Y, \Omega^1_{Y/\mathbb{C}}).$$

P3. Let X_0 be the affine plane curve defined by the equation

$$y^3 - 3y = x^5$$

over the complex numbers, and let X be the projective smooth model of X_0.

(a) Show that X_0 is nonsingular.

(b) Find all $a \in \mathbb{C}$ for which the polynomial $P_a(y) = y^3 - 3y - a$ has repeated roots. For each such a, factor the polynomial $P_a(y)$.

(c) Let $\pi : X \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ be the unique extension of the coordinate map $x : X_0 \longrightarrow \mathbb{A}^1_{\mathbb{C}}$. Describe the ramification divisor of π and compute its degree.

(d) Compute the genus of X by applying Hurwitz's theorem to $\pi : X \longrightarrow \mathbb{P}^1$.

A solution.

A1.

(a) Only X_2 among the three is nonsingular. The normalization map is a set-theoretic bijection in the case of X_1, but not in the case of X_3.

(b) Since X_1 embeds into the affine plane, the Zariski cotangent space at every $\mathbb{C}$-valued point of X_1 has dimension at most 2. At $(0,0,0) \in X_2$, the Zariski cotangent space has dimension 3.

(c) By Bézout's theorem, any two distinct irreducible curves on X_2 intersect; this is not the case of X_1, nor of X_3. The exceptional divisor on X_3 has self-intersection -1, while no prime divisor on X_1 has strictly negative self-intersection (one can translate any prime divisor into a *different* prime divisor, using the action of $PGL_2 \times PGL_2$ by fractional linear transformations).

A2.

(a) One has a short exact sequence of sheaves on P:

$$0 \longrightarrow \mathcal{O}_P(-3) \longrightarrow \mathcal{O}_P \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

hence the Hilbert polynomial is

$$h_X(T) = \binom{T+3}{3} - \binom{T+3-3}{3} = \frac{3}{2}T^2 + \frac{3}{2}T + 1.$$

(b) By the assumptions on f and g, they are relatively prime, and we have an exact sequence

$$0 \longrightarrow \mathcal{O}_P(-5) \longrightarrow \mathcal{O}_P(-2) \oplus \mathcal{O}_P(-3) \longrightarrow \mathcal{O}_P \longrightarrow \mathcal{O}_Y \longrightarrow 0,$$

hence the Hilbert polynomial is

$$h_Y(T) = \binom{T+3}{3} - \binom{T+1}{3} - \binom{T}{3} + \binom{T-2}{3} = 6T - 3$$

(c) By Serre duality it is equal to the arithmetic genus, or

$$1 - \chi(\mathcal{O}_Y) = 1 - h_Y(0) = 4.$$

A3.

(a) By the Jacobian criterion of smoothness, at a singular point we have

$$y^3 - 3y = x^5, \;\; 0 = 5x^4 \;\text{ and } 3y^2 - 3 = 0,$$

which has no solution.

(b) We have $P_a'(y) = 3(y^2 - 1)$, and

$$P_a(y) = \frac{y}{3}P_a'(y) - 2y - a,$$

so $P_a(y)$ is separable exactly when $a \neq \pm 2$. When $a = 2$, $P_a(y) = (y+1)^2(y-2)$ and when $a = -2$, $P_a(y) = (y-1)^2(y+2)$.

(c) For $x \in \mathbb{A}^1(\mathbb{C})$, the fiber $\pi^{-1}(x)$ consists of three distinct points when $x^5 \neq \pm 2$. When $x^5 = \pm 2$, the fiber consists of two points, one point P_x with multiplicity two and the other with multiplicity one.

Since 3 is prime to 5, $\pi^{-1}(\infty)$ consists of one point P_∞ with multiplicity three.

In this notation, the ramification divisor is

$$R = \sum_{x^5=\pm 2} 1 \cdot [P_x] + 2[P_\infty],$$

and has degree 12.

(d) Let g be the genus of X, and Hurwitz formula reads

$$2g - 2 = 3 \cdot (2 \cdot 0 - 2) + 12,$$

so $g = 4$.

PROPOSED QUESTIONS FOR QUALIFYING EXAMINATION IN ALGEBRAIC TOPOLOGY (2013 SPRING)

(1) Let $\mathbb{H}$ be the space of quaternions, and denote by $\mathbb{S}^3$ the unit sphere inside $\mathbb{H}$. The quaternion group $G = \{\pm 1, \pm i, \pm j, \pm k\}$ acts on $\mathbb{H}$ by left multiplication, and the action preserves the unit sphere $\mathbb{S}^3$. Let X be the quotient space $\mathbb{S}^3/G$. Compute its fundamental group $\pi_1(X)$ and its first homology group $H_1(X, \mathbb{Z})$.

($\mathbb{H}$ is spanned by four independent unit vectors $1, i, j, k$ as a real normed vector space. The multiplication between two elements of $\mathbb{H}$ is bilinear and is determined by the rules $i^2 = j^2 = k^2 = ijk = -1$, and 1 is the multiplicative identity.)

(2) Use Z to denote the subset of $\mathbb{R}^2$ that is given using standard polar coordinates (r, θ) by the equation $r = \cos^2(2\theta)$. The set Z is depicted in Figure 1.

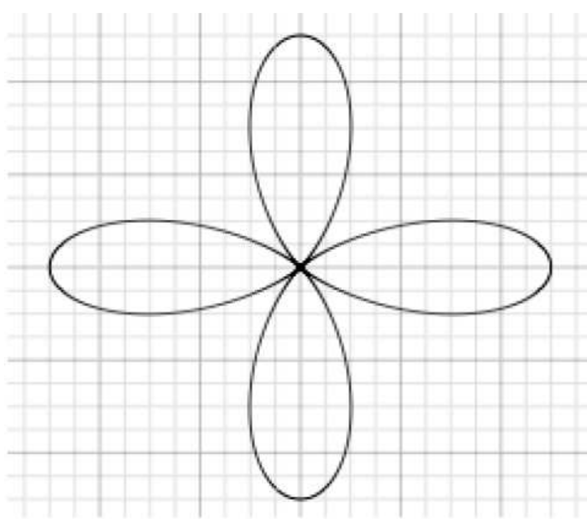

FIGURE 1. The set Z.

(a) Compute the fundamental group of Z.

(b) Let D denote the closed unit disk in $\mathbb{R}^2$ centered at the origin. The boundary of D is the unit circle, this denoted here by ∂D. Parametrize ∂D by the angle $\phi \in [0, 2\pi)$ and let f denote the map from the boundary of ∂D to Z that sends the angle ϕ to the point in Z with polar coordinates $(r = cos^2(2\phi), \theta = \phi)$. Let X denote the space that is obtained from the disjoint union of D and Z by identifying $\phi \in \partial D$ with $f(\phi) \in Z$. Give a set of generators and relations for the fundamental group of X.

(3) Let $K \subset \mathbb{R}^3$ denote a knot, this being a compact, connected, dimension 1 submanifold.
   (a) Compute the homology of the complement in $\mathbb{R}^3$ of any given knot K.
   (b) Figure 2 shows a picture of the trefoil knot.

FIGURE 2. The trefoil knot.

Sketch on this picture a curve or curves in the complement of K that generate(s) the first homology of $\mathbb{R}^3 - K$.
   (c) A Seifert surface for a knot in $\mathbb{R}^3$ is a connected, embedded surface with boundary, with the knot being the boundary (we do not impose orientability here). By way of an example, view the unit circle in the xy plane as a knot in $\mathbb{R}^3$. This is called the unknot. The unit disk in the xy plane is a Seifert surface for the unknot.
      (i) Compute the second homology of the complement in $\mathbb{R}^3$ of any given Seifert surface for the unknot.
      (ii) Sketch a Seifert surface for the unknot whose complement is not simply connected.

Proposed Answer

(1) Since G is a finite group and it acts on $\mathbb{S}^3$ freely, the quotient map $\mathbb{S}^3 \to X$ is a covering map. $\mathbb{S}^3$ is simply connected and hence it is the universal cover of X, where G acts as Deck transformations. Thus $\pi_1(X) = G$.

The first homology group is the abelianization of the fundamental group. Thus

$$\begin{aligned} H_1(X) &= \pi_1(X)/[\pi_1(X), \pi_1(X)] \\ &= G/[G,G] = G/\{\pm 1\} \\ &= \mathbb{Z}_2 \oplus \mathbb{Z}_2. \end{aligned}$$

(2) (a) The fundamental group is $\mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$.
  (b) The fundamental group is generated by a, b, c, d with $abcd = 1$.

(3) (a) $H_0 = \mathbb{Z}$, $H_1 = \mathbb{Z}$, $H_2 = \mathbb{Z}$, $H_3 = 0$ by Mayer Vietoris sequence.
  (b) A circle surrounding a segment of the knot.
  (c) (i) $H_2 = \mathbb{Z}$.
    (ii) A Mobius strip.

Exercise 1. The following questions are independent.
a) For any $a \in (-1, 1)$, compute
$$\int_0^{2\pi} \frac{\mathrm{d}t}{1 + a\cos t}.$$
b) For any $p > 1$, compute
$$\int_0^{\infty} \frac{\mathrm{d}x}{x^p + 1}.$$

Exercise 2. Is there a conformal map between the following domains? If the answer is yes, give such a conformal map. If it is no, prove it.
a) From $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ to $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$.
b) From the intersection of the open disks $D((0,0), 3)$ and $D((0,3), 2)$ to $\mathbb{C}$.
c) From $\mathbb{H}\backslash(0, i]$ to $\mathbb{H}$.
d) From $\mathbb{D}$ to $\mathbb{C}\backslash(-\infty, -\frac{1}{4}]$.

Exercise 3. The following questions are independent.
a) Describe all harmonic functions on the plane $\mathbb{R}^2$ that are bounded from above.
b) Let $h : \mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\} \to \mathbb{C}$ be holomorphic. Assume that $|h(z)| \leq 1$ for any $z \in \mathbb{H}$ and i is a zero of h of order $m \geq 1$. Prove that, for any $z \in \mathbb{H}$,
$$|h(z)| \leq \left|\frac{z - i}{z + i}\right|^m.$$

Solution of exercise 1.
a) The case $a = 0$ is obvious, we assume $a \neq 0$ from now. We have

$$\int_0^{2\pi} \frac{\mathrm{d}t}{1 + a\cos t} = \int_T \frac{\mathrm{d}z}{iz\left(1 + \frac{a}{2}\left(z + \frac{1}{z}\right)\right)} = \frac{1}{i}\int_T \frac{2\mathrm{d}z}{az^2 + 2z + a} = \frac{1}{i}\int_T \frac{2\mathrm{d}z}{a(z - z_1)(z - z_2)}.$$

where $z_1 = \frac{-1-\sqrt{1-a^2}}{a}$, $z_2 = \frac{-1+\sqrt{1-a^2}}{a}$. The point z_1 is outside the closed unit disk while the point z_2 is inside the open unit disk; thus the residue theorem leads to

$$\int_0^{2\pi} \frac{\mathrm{d}t}{1 + a\cos t} = 2\pi \mathrm{Res}\left(\frac{2}{a(z - z_1)(z - z_2)}, z_2\right) = \frac{2\pi}{\sqrt{1 - a^2}}.$$

b) For $z = re^{i\theta}$, $r > 0$, $-\pi < \theta < \pi$, let $\log z = \log r + i\theta$ The function $\frac{1}{1+x^p}$ is the restriction to $(0, +\infty)$ of $f(z) = \frac{1}{1+\exp(p\log z)}$, which is analytic in $\mathbb{C} - (-\infty, 0]$. For $0 < \epsilon < R < \infty$ we consider the positively oriented contour $\Gamma = \Gamma(\epsilon, R)$ made of the following four arcs:

- the closed interval $[\epsilon, R]$,
- the arc of circle of radius R and angle from 0 to $\frac{2\pi i}{p}$.
- the interval joining $Re^{i\frac{2\pi}{p}}$ and $\epsilon e^{i\frac{2\pi}{p}}$
- the arc of circle of radius ϵ and angle from $\frac{2\pi i}{p}$ to 0.

For $R > 1$ there is only one 0 of f inside the contour, $z_p = e^{i\frac{\pi}{p}}$. The residue at z_p can be computed to be

$$\mathrm{Res}(f, z_p) = -\frac{e^{i\frac{\pi}{p}}}{p}.$$

The residue theorem applied to Γ for $R \to \infty$ and $\epsilon \to 0$ allows to conclude that

$$\int_0^\infty \frac{\mathrm{d}x}{x^p + 1} = \frac{\pi}{p}\frac{1}{\sin(\pi/p)}.$$

Solution of exercise 2.
a) Yes, $i\frac{1+z}{1-z}$.
b) No. The inverse map would be a conformal map from the plane to a bounded set. It would therefore constant, by Liouville's theorem. This is absurd.
c) Yes, $\sqrt{z^2 + 1}$. To see this, $z \mapsto z^2$ sends the domain to $\mathbb{C}/[-1, \infty)$. Therefore $z^2 + 1$ sends the domain to $\mathbb{C}/[0, \infty)$. Taking the square root maps it to the upper half plane.
d) Yes, it can be deduced from questions a) and c), going from the disk to the upper half plane and then the complement of a ray. It is the Koebe function $\frac{z}{(1-z)^2}$.

Solution of exercise 3.
a) Let v be the harmonic conjugate of u. Then the function

$$H(z) = \exp(u(x, y) + iv(x, y)),\ z = x + iy,$$

is entire and bounded in the complex plane. By Liouville's theorem it is constant, which implies that u is constant.
b) The map $\Phi : z \mapsto i\frac{1+z}{1-z}$ is conformal from $\mathbb{D}$ to $\mathbb{H}$. Let $f(z) = h(\Phi(z))$.
The function $g(z) = f(z)/z^m$ is analytic on $\mathbb{D}$ and $|g(z)| \leq 1$ on $\partial\mathbb{D}$. The maximum principle implies that $|g(z)| \leq 1$ on $\mathbb{D}$ and therefore $|f(z)| \leq |z|^m$ for any z in $\mathbb{D}$. This implies the result.

DIFFERENTIAL GEOMETRY (QUALIFYING EXAMS, SPRING 2013)

(1) The Heisenberg group is the subgroup of $\mathrm{Sl}(3,\mathbb{R})$ composed of the 3×3, upper triangular matrices with 1 on the diagonal, this being the set of matrices of the form:

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{with} \quad (x,y,z)\in\mathbb{R}^3.$$

This group is observably diffeomorphic to $\mathbb{R}^3$.

(a) Compute the Lie algebra of the Heisenburg group.

(b) Exhibit a left-invariant Riemannian metric on the Heisenberg group.

(2) View $\mathbb{R}^2\times\mathbb{C}$ as the product complex line bundle over $\mathbb{R}^2$ and let θ_0 denote the connection on this line bundle whose covariant derivative acts on a given section s as ds with d being the exterior derivative. Let A denote the connection

$$\theta_0 + \frac{i}{1+x^2+y^2}(xdy - ydx).$$

(a) Compute as a function of $r\in(0,\infty)$ the linear map from $\mathbb{C}$ to $\mathbb{C}$ that is obtained by using A to parallel transport a given non-zero vector in $\mathbb{C}$ in the clockwise direction on the circle where $x^2+y^2=r^2$ from the point $(r,0)$ to itself.

(b) Give a formula for the curvature 2-form of the connection A.

(3) Use (t,x,y,z) to denote the Euclidean coordinates for $\mathbb{R}^4$. Let $t\mapsto a(t)$ denote a strictly positive function on $\mathbb{R}$. A Riemannian metric on $\mathbb{R}^4$ is given by the quadratic form:

$$g = dt\otimes dt + a(t)^2\Big(dx\otimes dx + dy\otimes dy + dz\otimes dz\Big).$$

Compute the components of the Riemann curvature tensor of g using the orthonormal basis $\{dt, adx, ady, adz\}$ for $\mathrm{T}^\star\mathbb{R}^4$.

Solutions

Question 1. We denote an element of the Heisenberg group as

$$M = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

(1.a). To identify the generators of the Lie algebra, we compute the left Maurer-Cartan one-form:

$$\theta_L = M^{-1}dM = dxX + dyY + (dz - xdy)Z,$$

where

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

are the generators of the Lie algebra. The Lie bracket is given by the commutators of these matrices:

$$[X, Z] = [Y, Z] = 0, \quad [X, Y] = Z.$$

(1.b). Denoting the transpose of a matrix U as U^t and its trace as Tr U , the following is trivially a left-invariant Riemannian metric:

$$g_L = \mathrm{Tr}(\theta_L \cdot \theta_L^t) = dx^2 + dy^2 + (dz - xdy)^2.$$

Question 2

(2.a). The parametric equation of the curve is

$$\gamma : [0, 2\pi] \to \mathbb{R}^2 : t \mapsto \gamma_t = (r \cos\ t, -r \sin\ t).$$

Pulling back the connection to the curve gives

$$\gamma_t^\star A = \gamma_t^\star \theta_0 - \frac{ir^2 dt}{1 + r^2}.$$

To find the parallel transport of $s_0 \in \mathbb{C}$ along the curve γ_t, we solve the first order differential equation with initial condition $s(0) = s_0$:

$$\nabla_{\dot\gamma_t} s(t) = 0,$$

where $\dot\gamma_t$ is $d\gamma_t/dt$ and the covariant derivative on the curve is computed with respect to the connection one-form $\gamma_t^\star A$:

$$\nabla_{\dot\gamma_t} = \frac{d}{dt} - \frac{ir^2 dt}{1 + r^2}.$$

With the initial condition $s(0) = s_0$, the solution is

$$s(t) = s_0 \exp\Big(\frac{t\ r^2}{1 + r^2} i\Big).$$

The parallel transport of s_0 is then given by evaluating $s(t)$ at the end point $t = 2\pi$. This defines the following linear map:

$$\mathbb{C} \to \mathbb{C} : s_0 \mapsto s_0 \exp\Big(\frac{2\pi r^2}{1 + r^2} i\Big).$$

(2.b). The curvature of the connection A is :

$$\Omega(A) = dA + A \wedge A = \frac{2i}{(1+x^2+y^2)^2} dx \wedge dy.$$

Question 3

We use the following notation:

$$e^0 = dt, \quad e^1 = adx, \quad e^2 = ady, \quad e^3 = adz, \quad \dot{a} = \frac{da}{dt}, \quad \ddot{a} = \frac{d^2a}{dt^2}.$$

The metric can then be rewritten as

$$g = \sum_{m=0}^{3} e^m \otimes e^m.$$

We can compute the connection from the Cartan's structure equations with zero torsion:

$$de^m + \sum_{n=0}^{3} \omega^m{}_n \wedge e^n = 0, \quad m = 0, 1, 2, 3.$$

Since the connection is compatible with the metric, $\omega^m{}_n$ has to be antisymmetric in m and n. we can then uniquely determine its components. A direct calculation gives:

$$de^0 = 0, \quad de^i = \frac{\dot{a}}{a} e^0 \wedge e^i, \quad i = 1, 2, 3,$$

from which we get:

$$\omega = \frac{\dot{a}}{a} \begin{pmatrix} 0 & -e^1 & -e^2 & -e^3 \\ e^1 & 0 & 0 & 0 \\ e^2 & 0 & 0 & 0 \\ e^3 & 0 & 0 & 0 \end{pmatrix}.$$

The Riemann curvature is then:

$$R(\omega) = d\omega + \omega \wedge \omega = \frac{1}{a^2} \begin{pmatrix} 0 & -a\ddot{a}\ e^0 \wedge e^1 & -a\ddot{a}\ e^0 \wedge e^2 & -a\ddot{a}\ e^0 \wedge e^1 \\ a\ddot{a}\ e^0 \wedge e^1 & 0 & -\dot{a}^2\ e^1 \wedge e^2 & -\dot{a}^2\ e^1 \wedge e^2 \\ a\ddot{a}\ e^0 \wedge e^2 & \dot{a}^2\ e^1 \wedge e^2 & 0 & -\dot{a}^2\ e^1 \wedge e^3 \\ a\ddot{a}\ e^0 \wedge e^3 & \dot{a}^2\ e^1 \wedge e^3 & \dot{a}^2\ e^2 \wedge e^3 & 0 \end{pmatrix}.$$

1. Suppose $f_j, j = 1, 2, \ldots$ and f are real functions on $[0,1]$. Define $f_j \to f$ in measure if and only if for any $\varepsilon > 0$ we have

$$\lim_{j\to\infty} \mu\{x \in [0,1] : |f_j(x) - f(x)| > \varepsilon\} = 0$$

where μ is the Lebesgue measure on $[0,1]$. In this problems, all functions are assumed to be in $L^1[0,1]$.

(a) Suppose that $f_j \to f$ in measure. Does it implies that

$$\lim_{j\to\infty} \int |f_j(x) - f(x)| \mathrm{dx} = 0.$$

Prove it or give a counterexample.

(b) Suppose that $f_j \to f$ in measure. Does this imply that $f_j(x) \to f(x)$ almost everywhere in $[0,1]$? Prove it or give a counter example.

(c) Suppose that $f_j(x) \to f(x)$ almost everywhere in $[0,1]$. Does it implies that $f_j \to f$ in measure? Prove it or give a counter example.

2.

(a) For any bounded positive function f define

$$A(f) := \int_0^1 f(x) \log f(x) \mathrm{d}x, \quad B(f) := \Big(\int_0^1 f(x) \mathrm{d}x \Big) \log \Big(\int_0^1 f(y) \mathrm{d}y \Big).$$

There are three possibilities: (i) $A(f) \geq B(f)$ for all bounded positive functions, (ii) $B(f) \geq A(f)$ for all bounded positive functions, and (iii) $A(f) \geq B(f)$ for some functions while $B(f) \geq A(f)$ for some functions. Decide which possibility is correct and prove your answer. If you use any inequality, state all assumptions of the inequality precisely and clearly.

(b) Let $\hat{f}$ denote the Fourier transform of the function f on $\mathbb{R}$. Suppose that $f \in C^\infty(\mathbb{R})$ and

$$\|\hat{f}(\xi)\|_{L^2(\mathbb{R})} \leq \alpha, \qquad \||\xi|^{1+\varepsilon} \hat{f}(\xi)\|_{L^2(\mathbb{R})} \leq \beta$$

for some $\varepsilon > 0$. Find a bound on $\|f\|_{L^\infty}(\mathbb{R})$ in terms of α, β and ε.

3. Assume that $X_1, X_2, \ldots$ are independent random variables uniformly distributed on $[0,1]$. Let $Y^{(n)} = n \inf\{X_i, 1 \leq i \leq n\}$. Prove that $Y^{(n)}$ converges weakly to an exponential random variable, i.e. for any continuous bounded function $f : \mathbb{R}^+ \to \mathbb{R}$,

$$\mathbb{E}\Big(f(Y^{(n)})\Big) \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^+} f(u) e^{-u} \mathrm{d}u.$$

Solutions:

1a. no. let $f = 0$ and

$$f_j(x) = j,\ 0 \le x \le 1/j; \quad = 0 \text{ otherwise}$$

1b. no. let $f = 0$ and f_j are the j-th Haar functions.

1c. yes. Fix any $\varepsilon > 0$. Let

$$E_j = \{x \in [0,1] : |f_j(x) - f(x)| > \varepsilon\}, \quad F_j = \cup_{k \ge j} E_j$$

By changing x on a set of measure zero, we have $f_j(x) \to f(x)$ for all x. Thus

$$\lim_{j \to \infty} F_j = \emptyset$$

Hence $\mu(F_j) \to 0$.

Alternatively, WLOG, we assume that $f = 0$. Let $g_j = \min(1, |f_j|)$. Then $g_j \to 0$. Thus

$$\mu(E_j) \le \varepsilon^{-1} \int_0^1 g_j(x) \mathrm{d}x \to 0.$$

2a. Since the function $x \to x \log x$ is convex, by Jensen inequality, we have $A(f) \ge B(f)$.

2b. From the Fourier inversion formula and Schwarz inequality,

$$|f(x)| \le \Big| \int \hat{f}(\xi) e^{2\pi i x \xi} \mathrm{d}\xi \Big| \le \int |\hat{f}(\xi)|^2 (1 + |\xi|)^{1+\varepsilon} \mathrm{d}\xi \int (1 + |\xi|)^{-1-\varepsilon} \mathrm{d}\xi \le \varepsilon^{-1} C[\alpha^2 + \beta^2]$$

for some constant C.

3. Up to a permutation, we can assume that $x_1 \le x_2 \ldots \le x_n$. In this case, $Y^{(n)} = n x_1$. Thus

$$\mathbb{E} f(Y^{(n)}) = n! \int_0^1 \mathrm{d}x_1 \int_{x_1}^1 \mathrm{d}x_2 \ldots \int_{x_{n-1}}^1 \mathrm{d}x_n f(n x_1)$$

$$= n! \int_0^1 \mathrm{d}x_1 \int_{x_1}^1 \mathrm{d}x_2 \ldots \int_{x_{n-2}}^1 \mathrm{d}x_{n-1} (1 - x_{n-1}) f(n x_1)$$

$$= n \int_0^1 \mathrm{d}x_1 f(n x_1)(1 - x_1)^{n-1} = \int_0^n \mathrm{d}x\, f(x)(1 - x/n)^{n-1} \to \int_{\mathbb{R}^+} f(u) e^{-u} \mathrm{d}u.$$

where we have used dominated convergence in the last step.