Sample Questions from Past Qualifying Exams

This list may give the impression that the exams consist of a series of questions fired at the student one after another. In fact most exams have more the character of a conversation with considerable give and take. Hence this list cannot be expected to indicate accurately the difficulties involved.

The list indicates the professor associated to each question where available. Some have been in the MGSA files for a while, and this information has been lost (if it was ever there).

The listing by section is approximate, since some questions may fit under more than one heading.

Also available from the MGSA is an extended syllabus and sample Qual questions with answers that Danny Calegari wrote. The questions on Complex Analysis are included here, but his write-up includes solutions as well.

Complex Analysis

- Given a function continuous in a disk and analytic everwhere but at the center, prove that the function is analytic in the entire disk.

- Give a proof of Picard's theorem using, for example, the fact that the j invariant of a modular curve uniformizes the $2,3,\infty$ hyperbolic triangle as the upper half-plane.

- Show that the mapping group of the torus is $SL(2,\mathbb{Z})$.

- Let $\Omega = \mathbb{C}\backslash\{x \in \mathbb{R} \mid x < \frac{1}{4}\}$. Is there a conformal isomorphism $f\colon \Delta \to \Omega$, where Δ is the open unit disk? [**McMullen**]

- Is there one with $f(0) = 0$? [**McMullen**]

- How can we arrange for a unique f with $f(0) = 0$? [**McMullen**]

- What can you say about the coefficients a_i of the power series expansion $f = \sum a_j z^j$? [**McMullen**]

- For f with $f'(0) \in \mathbb{R}$, $f'(0) \geq 0$, what ring do the a_i lie in? [**McMullen**]

- So to show that the a_i lie in this ring, can we write down another function in terms of f and −'s which maps Δ to Ω? [**McMullen**]

- Calculate a_i. Now what ring do the a_i lie in? [**McMullen**]

- How would you write down the power series for $\tan z$? [**McMullen**]

  - What is its radius of convergence? [**McMullen**]

- Can you prove what the zeroes of $\cos z$ are? [**McMullen**]

  - Why does the radius of convergence correspond this way? [**McMullen**]

- What is the area of a spherical triangle? Can you prove it? [**McMullen**]

- Same for hyperbolic triangle. [**McMullen**]

- Define a complex torus. [**McMullen**]

- What is the automorphism group of a complex torus? [**McMullen**]

- Show that if all the zeroes of a polynomial lie in a half-plane, then all zeroes of the derivative lie in the same half plane.

- What is the area of a spherical triangle?

- What is the automorphism group of a complex torus?

- If f_n is a family of holomorphic functions such that $f_n \to F$ uniformly on compact subsets of some domain Ω, what can you say about f'_n?

- Give an example of a sequence $f_n \to f$ where every f_n is holomorphic and injective, and f is not. Is this the most general such example?

- Why is there no conformal automorphism from the punctured disk to an annulus?

- Show that for a doubly periodic function f the number of zeroes of f and the number of poles of f (counting with multiplicities) is equal.

- Suppose f_i are harmonic functions on the unit disk D. Show that no linear combination of the f_i can be negative on ∂D and positive at some point in the interior of D.

- Find the poles and residues of $1/\sin(z)$.

- Give the formula for a conformal map from the unit disk to the inside of a polygon with angles $2\pi - \beta_i \pi$.

- Show that a continuous real-valued function u on some region Ω which has the mean-value property is harmonic.

- Suppose f is an analytic map from the punctured disk to $\mathbb{C}$. Can you write a power series expansion for f? What general form does it have? [**McMullen**]

- What can you say about the growth of the a_n's?

- Relate this to radii of convergence.

- If f is bounded, what additional things can you say?

- How would you compute the integral

$$\int_0^\infty \frac{x^{\frac{1}{2}}}{1+x^2}\, dx \quad ? \qquad [\textbf{McMullen}]$$

- Is the top half of a disk conformally isomorphic to the whole disk? What is the isomoprhism? [**McMullen**]

- What is the argument principle? [**Sarason**]

- Why is it called the argument principle? [**Sarason**]

- Generalize the argument principle to a statement about an arbitrary continuous function f from a domain to $\mathbb{C}$. You can assume f has isolated zeroes. [**Sarason**]

- You don't want to calculate any integrals do you? [**Sarason**]

- How do you prove the uniqueness part of the Riemann Mapping Theorem? [**Sarason**]

- What are the conformal automorphisms of the disk? [**Sarason**]

- What are the conformal automorphisms of the upper half plane? [**Sarason**]
- What is the modular group? [**Sarason**]
- Let f be holomorphic in Δ^*, the punctured unit disk, and suppose that

$$|f(z)| \leq \frac{1}{\sqrt{|z|}}$$

  Show that the singularity at 0 is removable. [**McMullen**]
- What is the Riemann zeta function? [**Poonen**]
- State the Riemann hypothesis. [**Poonen**]
- What is analytic continuation? Why is it unique?
- Does every complex analytic function have a power series?