Uniformization of Riemann Surfaces

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April 5, 2004

Abstract

The uniformization theorem states that every simply connected Riemann surface is conformally equivalent to the open unit disk, the complex plane, or the Riemann sphere. We present three aproaches to the uniformization of Riemann surfaces. We first prove the uniformization theorem via the construction of a harmonic function by the Dirichlet principle. We then give an alternate proof by triangulating the surface and inductively constructing an analytic map. Finally, we introduce projective structures on the surface and describe the geometric realization as a uniformizing map.

Contents

1 Introduction

At its roots, uniformization theory is closely tied to the formation of the concept of the Riemann surface. Riemann conceived the idea of the Riemann surface to deal with multivalued functions. He constructed such surfaces by pasting together sheets of the complex plane. Weierstrass, on the other hand, developed the idea of analytic continuation to deal with complex functions. Consider an analytic function u of a complex variable z. Such a function can be represented locally near a point z_0 by two analytic functions $z(t), u(t)$ of a local parameter t in the plane. These locally defined functions represent a function element. By analytically continuing the function element to all other possible function elements, and then adding further elements at singularities (branch points), we obtain an analytic form (z, u). Considering the function elements as points, the analytic form is the Riemann surface of the analytic function. In this formulation, the link between Riemann surface and analytic function is explicit. Weyl laid out the idea of the abstract Riemann surface, purely geometric and independent of the analytic function, by considering it as a manifold. After Weyl, we will take the following modern definition of a Riemann surface:

Definition 1.1. A *Riemann surface* is a connected Hausdorff space M together with a collection of charts $\{U_\alpha, z_\alpha\}$ with the following properties:

1. The U_α form an open covering of M.
2. Each z_α is a homeomorphic mapping of U_α onto an open subset of the complex plane $\mathbb{C}$.
3. If $U_\alpha \cap U_\beta \neq 0$, then $f_{\alpha\beta} = z_\beta \circ z_\alpha^{-1}$ is complex analytic on $z_\alpha(U_\alpha \cap U_\beta)$.

We will likewise consider uniformization theory in more modern terms, but the motivation of uniformization lies with Riemann and Weierstrass. As Weyl states, "in the theory of uniformization the ides of Weierstrass and of Riemann grow into a complete unity" (Weyl [10]). Weierstrass could represent his analytic form locally by a single-valued parameter varying in the complex plane, while Riemann could parametrize the form by considering z and u as functions of a point on a surface. From this perspective, the problem of uniformization is to obtain a *global* representation $z = z(t), u = u(t)$ of the analytic form in terms of the uniformizing parameter t whose domain is a subset of the complex plane. The parameter is uniform in that $z(t)$ and $u(t)$ are single-valued, and t must serve as a local parameter.

Leaving behind the analytic form, we would like to find a parameter t that varies in a domain R of the complex plane that is both a local and global parameter of an arbitrary Riemann surface M. This amounts to finding a map from R to M that is *analytic*. A great aid to us and those who first developed uniformization is Schwarz's idea of the covering surface. Every Riemann surface M has a universal covering surface $\widetilde{M}$ that is also a Riemann surface. If $f : R \to \widetilde{M}$ is a uniformizing map for $\widetilde{M}$ and $\pi : \widetilde{M} \to M$ is the covering map, we can easily get a uniformization of M by composing $\pi \circ f$. Since the universal covering surface is simply connected, we focus our attention on simply connected Riemann surfaces. To get the strongest uniformization, our map from R onto $\widetilde{M}$ must be one-to-one, in which case R must be simply connected as well. In geometric terms, we want a conformal (angle-preserving) map between $\widetilde{M}$ and a domain R of the complex plane. Allowing for compact simply connected Riemann surfaces, we allow R to possibly be the whole Riemann sphere. $\widetilde{M}$ and R are then conformally equivalent, which means they are essentially the same.

Before coming to the full statement of the uniformization theorem, we mention a special case that applies to the simplest Riemann surfaces, those which are subsets of the complex plane.

Theorem 1.1 (Riemann mapping thoerem). *For any simply connected region R in the complex plane that is not the whole plane and $z_0 \in R$, there exists a unique conformal mapping f of R onto the unit disk such that $f(z_0) = 0$ and $f'(z_0) > 0$.*

The theorem may have been suggested to Riemann by physical considerations of fluid flow or electric fields in such domains, for he made use of the Dirichlet principle; we will discuss this idea further in section 2. The Riemann mapping thoerem groups all simply connected regions except for the whole plane into one conformal class, that of the unit disk. The theorem will be an important tool in one of our proofs of the unformization theorem.

Returning to general Riemann surfaces, we now state the uniformization theorem that we will prove in this thesis:

The Uniformization Theorem. *Every simply connected Riemann surface is conformally equivalent to the unit disk, the complex plane, or the Riemann sphere.*

The uniformization theorem was first proved by Koebe and Poincaré independently in 1907. It is a classification theorem of all Riemann surfaces according to their universal covering spaces into three groups. Importantly, it reduces many aspects of Riemann surfaces to the study of the disk, the plane, and the sphere.

It is easy to see that these three spaces are indeed conformally distinct. Compactness distinguishes the sphere from the disk or the plane topologically. As for distinguishing the disk from the plane, we can never have a conformal map from the plane onto the disk, for such a map would be a bounded entire function and hence constant by Liouville's thoerem.

As the Riemann surface evolved, so did approaches to uniformization. In an attempt to convey a sense of the variety of ideas that have been related to uniformization, we will present three approaches to the problem. The first has already been mentioned in connection with the Riemann mapping thoerem. That is, we will use physics as motivation to prove the uniformization theorem using harmonic functions and Dirichlet integrals. The Dirichlet principle was fully justified by Hilbert, and the proof given here is based on his 1909 proof of uniformization. The logic will largely follow Siegel [9]; the proof by Weyl [10] is similar. Following our first proof of the uniformization theorem, we will immediately apply the theorem to arbitrary Riemann surfaces; the perspective is mainly from Ahlfors [3]. The second approach will also use classical methods, but will deal strictly with analytic functions and involve the triangulation of a Riemann surface to prove the uniformization theorem. The main tools will be the Schwarz reflection principle, the Riemann mapping theorem, and a statement derived from Koebe distortion theorem. The proof follows that of Sansone and Gerretsen [8]. The third approach is distinctly more abstract, making use of more recent tools of cohomology. There we will present projective structures on Riemann surfaces and their geometric realizations, following Gunning [6] [7]. However, by this approach we are not able to provide a full proof of the uniformization theorem. We consider briefly some possible resolutions to the problem. This final approach provides an alternate perspective on uniformization and serves as an introduction to projective structures and more modern methods of studying of Riemann surfaces.

2 The First Proof

2.1 Preliminaries

Our goal is to construct a global analytic function from a simply connected Riemann surface to a subdomain of the Riemann sphere. How do we get an analytic function on the surface? We are aided by thinking of analytic functions as conformal, for physics provides us with natural conformal maps. If we place a system of charges on a Riemann surface and keep the boundary (if any) grounded, the electric field lines and equipotential lines on the surfaces must be perpendicular at every point. This suggests that if we map the former to horizontal lines in the plane, and the latter to vertical lines, the map will be conformal. We let u denote the potential function.

Given a set of charges, how do we get the potential function in a region R? The potential function is that which minimizes the total field energy. This energy is given by the Dirichlet integral:

$$D[u; R] = \int_R (u_x^2 + u_y^2) dx dy. \tag{2.1}$$

For later, we will also need to define the Dirichlet inner product of two functions u and v:

$$D[u, v; R] \int_{\Omega} (u_x v_x + u_y v_y) dxdy. \tag{2.2}$$

In fact, the function that minimizes the Dirichlet integral is a *harmonic function*, defined as having $\nabla^2 u = 0$, where ∇^2 is the Laplacian operator. From a harmonic function u, we can get a harmonic conjugate v, and if the conjugate is single-valued, we obtain an analytic function $f = u + iv$. From a mathematics standpoint, the problem we have been discussing is the Dirichlet problem: find a harmonic function that satisfies certain given boundary conditions. If such a solution exists, we can get it by minimizing the Dirichlet integral.

The next step is to figure out what boundary conditions will give a function that maps the Riemann surface to a domain in the sphere. If we have no boundary conditions, corresponding physically to no charges on our surface, then the potential function is constant. So we must place some charge on our surface. The simplest nontrivial boundary condition would be to require a simple pole $1/z$ for our function, corresponding to a dipole charge. Actually, we could specify just one charge, corresponding to the function $log|z|$. The potential function solving this boundary problem is Green's function, and this situation can be used in the proof of the uniformization theorem (see Ahlfors [3]). However, Green's function does not exist on some Riemann surfaces, so we will not consider it here.

Why should we expect a function with a simple pole to give us the correct map? On a sphere, it obviously maps to the whole sphere. On the plane, consider the pole at the origin as a dipole oriented in the positive x-direction. For all fixed values $v = c$, except for one, tracing an electric field flow line on the plane traces a circle tangent to the x-axis. Only tracing the flow line on the x-axis itself gives a nonclosed path, for this axis extends to a point at infinity on either side.

Now consider the dipole on a simply connected region that is not the plane or sphere, and let the boundary be grounded; this is the case of the Riemann mapping theorem. Then again we expect the flow lines to be closed, except for one line $v = c$ which extends out to the boundary, forks on the boundary, and comes together and returns from a different point on the boundary. Mapping such lines onto the plane, the dipole corresponds to ∞, and all of the plane is covered, except for the line $v = c$, which has a segement covered twice, coresponding to the fork of the boundary. Thus we have a plane with a slit. The slit can be mapped by a conformal one-to-one map to the negative real half-axis; the resulting region can be mapped conformally and one-to-one to the right half plane, and then to the unit disk.

These three cases considered correspond to the three cases for Riemann surfaces, and they motivate the method of proof that we give here, which will be in two main parts. Given a simply connected Riemann surface, we will construct the analytic map from the surface to a domain in the sphere. To do this, we will find the harmonic function that is the real part of a function with a simple pole by minimizing Dirichlet integrals; taking the harmonic conjugate gives us the analytic map. Then we will prove the bijectivity of the map, and show that the image of a Riemann surface under the map is conformally equivalent to the sphere, the plane, or the unit disk.

2.2 Construction of the Analytic Map

Throughout this section, M will denote the Riemann surface in question. By disk we will mean a closed disk, and if R is a disk, then $\underline{R}$ will denote the interior of the disk.

2.2.1 The Cover

Before beginning our construction, we need two statements concerning the covering of our Riemann surface by disks. The first statement, the (second) axiom of countability, is actually assumed for topological surfaces by Weyl [10], as is the finiteness condition for compact surfaces.

Both statements can be proved from our basic defintion 1.1 for Riemann surfaces (see Ahlfors [4]).

Proposition 2.1. *Every Riemann surface M can be covered with countably many disks belonging to M, so that every point of M is in the interior of one of the covering disks.*

Proposition 2.2. *On every Riemann surface M it is possible to choose finitely many disks $R_1, \ldots, R_\nu$ or countably many disks $R_1, R_2, \ldots$ which cover M, in the sense of proposition 2.1, such that for $n = 1, \ldots, \nu - 1$ and $n = 1, 2, \ldots$, respectively, the union of $\underline{R}_1, \ldots, \underline{R}_n$ intersects $\underline{R}_{n+1}$ without containing it. The number of required disks is finite if and only if M is compact.*

Now we may begin our construction. We choose a point p_0 in M and disks, as in proposition 2.2, such that $R_0 \subset R_1$ are both concentric about p_0, and such that $R_0 \cap (\cup_{i>2} R_i) = \emptyset$. Also, let $B_n = \cup_{i=0}^{n} R_i$, and let $C_0, C_1, \ldots$ be the boundaries of the disks $R_1, R_2, \ldots$. For any region R containing p_0, $\dot{R} = R \backslash \{p_0\}$.

We take a moment to define piecewise differentiable or harmonic functions and their Dirichlet integrals, with respect to our chosen cover. Real-valued functions that are continuous, differentiable, or harmonic on a region R of M have their usual definitions in terms of local parameters. A continuous function is *piecewise* differentiable on R if it has continuous first-order partial derivatives on $\dot{R}$, with the possible exception of points of finitely many circumferences C_n. A similar definition works for piecewise harmonic functions. To define the Dirichlet integral $D[h; R]$ of a piecewise differentiable function h in a region R bounded by circular arcs, we partition the region into nonoverlapping subregions bounded by circular arcs on the interior of which the function h is continuously differentiable. If the Dirichlet integrals converge on every subregion, their sum is the Dirichlet integral $D[h; R]$ over the whole region R. The definition is similar for the Dirichlet inner product $D[h, g; R]$.

2.2.2 Comparison Functions

Our goal is to construct a harmonic function that is the real part of a meromorphic fuction that has a simple pole at p_0 and is otherwise regular on M. Let a be the radius of the disk R_0, and let $z = x + iy$ be the local coordinate at p_0, with $z(p_0) = 0$. We define in R_0 the function

$$q = \mathrm{Re}(\frac{1}{z} + \frac{z}{a^2}) = \frac{x}{x^2 + y^2} + \frac{x}{a^2} \tag{2.3}$$

in terms of the local coordinate. The first term in q is the simple pole; the second term is to make the normal derivative of q equal to zero; we will need this property later. We want to consider functions h that look like q near p_0. More precisely, we require that the difference $h(p) - q(p)$ is continuous at p_0. This condition fixes the pole at p_0 as our boundary condition.

We would like to minimize the Dirichlet integral of these functions h to get our harmonic function. However, over any domain containing p_0, the Dirichlet integral of h diverges, so we need to modify the integral we use. Let us define

$$\hat{h} = \begin{cases} h - q & \text{for } p \text{ in } R_0 \\ h & \text{for } p \text{ in } M - R_0 \end{cases}$$

so that $\hat{h}$ in M is discontinuous on C_0. Let us also require the existence of $D[h - q; \dot{\underline{R}}_0]$, which we denote for convenience by $D[h - q; R_0]$. Then we can consider the modified integral

$$D[\hat{h}; B_n] = D[h; B_n - R_0] + D[h - q; R_0].$$

This modified integral, if it exists for $n = 1, 2, \ldots$, gives a monotonic increasing sequence of nonnegative numbers. If the sequence is bounded and has a limit, then we let

$$D[\hat{h}; M] = \lim_{n \to \infty} D[\hat{h}; B_n].$$

This is the *normalized Dirichlet integral* of h over M. So the functions h that we consider should have finite normalized Dirichlet integral.

The class of functions that we want to consider will be called *comparison functions*. To summarize the above discussion, h is a comparison function if it has the following three properties:

1. h is continuous and piecewise differentiable in $\dot{M}$.
2. $h - q$ is continuous at p_0.
3. $D[\hat{h}; R]$ exists.

Using polar coordinates $x = r\cos\phi, y = r\sin\phi$ on R_1 it can be verified that the function

$$h = \begin{cases} \frac{x}{x^2+y^2} = \frac{\cos\phi}{r} & \text{on } \dot{R}_0 \\ \frac{\cos\phi}{r}\frac{b-r}{b-a} & \text{on } R_1 - R_0 \\ 0 & \text{on } M - R_1 \end{cases}$$

has the required properties of a comparison function, so we know that comparison functions do exist.

The problem is now to get a comparison function that minimizes $D[\hat{h}; R]$. Since $D[\hat{h}; R] \geq 0$, the set of all comparison functions has some nonnegative greatest lower bound μ for the normalized Dirichlet integral. Hence there exists a sequence h_n such that $D[\hat{h}_n; R] \to \mu$ as n goes to ∞. Such a sequence will be callsed a *minimizing sequence*. Our main result is the solution of this given Dirichlet problem, i.e., the proof that μ is actually obtained for some comparison function, and that the comparison function is indeed harmonic. It is the harmonic function that will be the real part of our required conformal map.

2.2.3 Existence of the Minimal Comparison Function

A common method of demonstrating the existence of a function is to construct a convergent sequence of functions whose properties we know. In our case, we want our convergent sequence to be a minimizing sequence. We will need the following lemma for minimizing sequences; the proof is straightforward and is omitted.

Lemma 2.1. *For a minimizing sequence h_j,*

$$\lim_{m,n\to\infty} D[h_m - h_n; M] = 0. \tag{2.4}$$

Beginning with a minimizing sequence, our procedure will be to the functions of this sequence, eventually arriving at convergence to a harmonic function. Note that we have consructed our cover out of disks, so that, using the Poisson integral, we can easily modify our functions to be harmonic in a given disk and still continuous on the boundary. The role of the Poisson integral is that, in short, it solves the Dirichlet problem in a disk having continuous boundary values. Convergence of these disks can be obtained under certain conditions, as specified in the following lemma (Siegel [9]):

Lemma 2.2. *Let $u_n(z), n = 1, 2, \ldots$ be a sequence of functions harmonic in a Jordan domain $\mathfrak{G}$ and having Dirichlet integrals. Further, let $\lim_{n,m\to\infty} D[u_n - u_m; \mathfrak{G}] = 0$. If the sequence of functional values $u_n(z_0)$ converges at a point z_0 of $\mathfrak{G}$, then $u_n(z)$ coverges at all points z in $\mathfrak{G}$ to a limit function harmonic in $\mathfrak{G}$, and this convergence is uniform on every region in $\mathfrak{G}$.*

Our procedure will inductively get convergence to harmonic functions on each disk of our cover, in the prescribed order. While getting convergence on these successive disks, we want to make sure that the limit function does not change inside previous disks. In other words, the limit function should be the harmonic continuation of the limit function defined on previous disks. Our tool will be the Schwarz reflection principle for harmonic functions; from this principle one can derive the following lemma that will be used in the proof:

Lemma 2.3. *Let $u_n (n = 1, 2, \dots)$ be a sequence of functions harmonic inside a region B and having Dirichlet integrals. Let L be a circular arc which is a portion of the boundary of B, and let each function u_n vanish on L. Further, let $\lim_{n\to\infty} D[u_n; B] = 0$. Then $u_n \to 0$ in the interior of B, and this convergence is uniform on every region in B.*

Now we can give the key result.

Theorem 2.1. *There exists a comparison function u that is harmonic in $\dot{M}$ and satisfies*

$$D[\hat{u}; M] = \mu.$$

Proof. The proof will be carried out in 4 steps.

I. We first construct a minimizing sequence of functions harmonic in $\dot{\underline{R}}_1$.

Let h be a comparison function. Since $h - q$ is continuous on R_1, in particular on the boundary C_1, there is exactly one function, given by the Poisson integral, which is harmonic in $\underline{R}_1$, continuous on R_1, and coincident with $h - q$ on C_1. In $\dot{\underline{R}}_1$, denote this function by $g - q$. If we set $g = h$ in $R - R_1$, then g is continuous in all of $\dot{R}$ and is a comparison function.

We claim that the normalized Dirichlet integral of g is no greater than that of h. We must show that

$$D[g; R_1 - R_0] + D[g - q; R_0] \leq D[h; R_1 - R_0] + D[h - q; R_0].$$

Since $g - q$ is harmonic in $\underline{R}_1$, we have

$$D[g - q; R_1] \leq D[h - q; R_1]$$

which implies

$$\begin{aligned} D[g; R_1 - R_0] - 2D[q, g; R_1 - R_0] + D[g - q; R_0] \\ \leq D[h; R_1 - R_0] - 2D[q, h; R_1 - R_0] + D[h - q; R_0]. \end{aligned} \quad (2.5)$$

So the proof of our claim is reduced to showing that $D[q, g - h; R_1 - R_0] = 0$. Let B be the ring $R_1 - R_0$ bounded by the circles C_1 and C_0. Since q has derivatives of all orders in the interior of B, and these derivatives have boundary values, and since $g - h$ is continuous and piecewise differentiable and has a convergent Dirichet integral over B, we can use Green's Identity to get

$$\begin{aligned} D[q, g - h; R_1 - R_0] + \iint_{R_1 - R_0} (\Delta q)(g - h)dxdy \\ = \int_{C_1} (g - h)(q_x dy - q_y dx) - \int_{C_0} (g - h)(q_x dy - q_y dx). \end{aligned} \quad (2.6)$$

The second double integral vanishes because q is harmonic. The first line integral vanishes because g and h coincide on the boundary C_1. We can rewrite the second integral in terms of the normal derivative,

$$\int_{C_0} (g - h)(q_x dy - q_y dx) = \int_{C_0} (g - h)\frac{\partial q}{\partial n}|dz|,$$

and since, as noted above, the normal derivative of q is zero on C_0, this integral vanishes as well, and the claim is proved.

We start with any minimizing sequence. Replacing each member h of the sequence by the corresponding function g gives us a new minimizing sequence whose Dirichlet integrals converge to μ at least as well as h_n. Hence we may assume that for all h_n of a minimizing sequence, $h_n - q$ are harmonic in $\underline{R}_1$ and have boundary values throughout C_1. Adding a constant to a function does not change the Dirichlet integral, so we may assume that the $h_n - q$ all vanish at p_0. Such a minimizing sequence will be called a *smooth minimizing sequence*.

II. The next step is to show that for a smoothed minimizing sequence, the differences $h_n - q$ converge in $\underline{R}_1$ to a harmonic function u_0. From equation (2.4), we have

$$\lim_{m,n\to\infty} D[(h_m - q) - (h_n - q); R_1] = 0.$$

Using lemma 2.2, since the $h_n - q$ vanish at p_0 and are harmonic in $\underline{R}_1$, we obtain

$$\lim_{m,n\to\infty} (h_n - q) = u_0$$

in $\underline{R}_1$. The convergence is uniform in the interior of every circle smaller than and concentric with C_1, and u_0 is harmonic in $\underline{R}_1$. Then $u = u_0 + q$ is harmonic in $\underline{\dot{R}}_1$ and has the same singularity at p_0 as q, since $u_0(p_0) = 0$.

III. In general, it is not true that the smoothed sequence h_n converges in $\dot{R}$ outside of $\underline{\dot{R}}_1$. However, the following proof by induction shows that we can get a unique harmonic continuation of the function u, constructed in part II, to all points in $\dot{R}$.

Let t be a given natural number. Assume the existence of a smoothed minimizing sequence h_{tn} which converges uniformly in every region in $\underline{\dot{B}}_t$. By assumption, u is the harmonic continuation fo the function constructed on $\underline{\dot{R}}_1$. Further, let h_{tn} be piecewise harmonic on $\underline{\dot{B}}_t$, i.e., harmonic at all points in $\dot{B}_t$ not belonging to any of $C_1, \ldots, C_t$. A minimizing sequence with these properties is a t times smoothed minimizing sequence.

We have already constructed the 1 times smoothed minimizing sequnce. For the induction step, let us construct a $(t+1)$ times smoothed minimizing sequence from a given t times smoothed minimizing sequence. The disk R_{t+1} lies outside of R_0 and intersects B_t. Using the local uniformizing parameter, for each n we can form the Poisson integral with boundary values on C_{t+1} given by the function h_{tn}; this will define the function j_{tn} on R_{t+1}; this function is harmonic on $\underline{R}_{t+1}$. Outside of R_{t+1}, define $j_{tn} = h_{tn}$. Then since the Poisson integral minimizes the Dirichlet integral for given boundary values, we have

$$D[j_{tn}; R_{t+1}] \leq D[h_{tn}; R_{t+1}]$$

and therefore

$$D[j_{tn}; M] \leq D[h_{tn}; M].$$

Thus j_{tn} form a minimizing sequence of comparison functions.

Now we show that j_{tn} is $t+1$ times smoothed. Since, by assumption, h_{tn} is piecewise harmonic on $\underline{\dot{B}}_t$, it follows that j_{tn} is piecewise harmonic on $\underline{\dot{B}}_{t+1}$. In the intersection of R_{t+1} and B_t, choose a region B which abuts C_{t+1} along an arc L, but does not meet any of the circles $C_1, \ldots, C_t$. Then let $u_n = u + tn = h_{tn} - j_{tn}$. Since the combined sequence h_{tn}, j_{tn} is also a minimizing sequence, equation 2.4 implies

$$\lim_{n\to\infty} D[u_n; M] = 0 \qquad \lim_{n\to\infty} D[u_n; B] = 0 \tag{2.7}$$

Then we can use 2.3 to get

$$\lim_{n\to\infty} u_{tn} = 0$$

in the interior of B. By our induction assumption

$$\lim_{n\to\infty} h_{tn} = u$$

on B. On the other hand, as in part II, we see that j_{tn} converges uniformly to a harmonic function in the interior of every smaller circle concentric with C_{t+1}. Therefore, since the limit function coincides with u in the interior of B, it must be the harmonic continuation of u to all of $\underline{R}_{t+1}$. It is clear that the harmonic continuation is independent of the choice of arc L and domain B. To complete the induction argument, we set $h_{t+1,n} = j_{tn}$.

This inductive procedure allows us to define the function u successively on $\dot{R}_1, R_2, \ldots$ in a unique manner such that u is harmonic in each domain. Since every point of $\dot{M}$ lies in one of these domains, it follows that u is uniquely determined and harmonic on $\dot{M}$.

IV. We have our harmonic function, but we do not know that it is a comparison function, and we have not calculated its Dirichlet integral. So our last step is to show that the function does in fact u minimize the normalized Dirichlet integral. Since u is harmonic on $\dot{M}$, it is continuous and piecewise differentiable on $\dot{M}$. Further, the difference $u - q$ is continuous at p_0 and has the value 0 there. We show that the normalized Dirichlet integral of u converges to a value $\leq \mu$. We will need to use the convergence of sequences on compact subsets of M, so we diminish the region B_m by omitting from each of the covering disks $R_1, \ldots, R_m$ a boundary strip of width ϵ. Let $B_m(\epsilon)$ denote the union of the resulting disks $R_1(\epsilon), \ldots, R_m(\epsilon)$. By part II, $h_n - q$ converges uniformly to u on any $R_1(\epsilon)$, and the $h_n = h_{1n}$ form a smoothed minimizing sequence; by part III, $h_{t+1,n}$ converges uniformly to u on any $R_{t+1}(\epsilon)$, for any $\epsilon > 0$. On any such disk, this sequence is a convergent $t+1$ times smoothed minimizing sequence, and the $h_{t+1,n}$ coincide with the h_{tn} on $\dot{M}$ outside $\underline{R}_{t+1}$. This is true for $t = 1, \ldots, m-1$, so by Harnack's theorem for harmonic functions, we have

$$\lim_{n\to\infty} D[\hat{h}_{mn}; B_m(\epsilon)] = D[\hat{u}; B_m(\epsilon)].$$

Since h_{mn} is a minimizing sequence,

$$\lim_{n\to\infty} D[\hat{h}_{mn}; M] = \mu.$$

On the other hand,

$$D[\hat{h}_{mn}; B_m(\epsilon)] \leq D[\hat{h}_{mn}; M],$$

and therefore

$$D[\hat{u}; B_m(\epsilon)] \leq \mu;$$

μ is independent of ϵ and m. Letting $\epsilon \to 0$ and $m \to \infty$, we get

$$D[\hat{u}; M] \leq \mu.$$

Then u is a comparison function, which implies

$$D[\hat{u}; M] \geq \mu,$$

and finally

$$D[\hat{u}; M] = \mu.$$

□

2.2.4 Uniqueness of the Minimal Function

We have done the main work of finding our minimizing harmonic function. In the following section, we prove, in succession, that the function u is unique for a fixed cover $R_0, R_1, \ldots$; independent of the choice of R_0 when $R_1, R_2, \ldots$ are fixed; and independent of the choice of disks $R_1, R_2, \ldots$, so long they satisfy proposition 2.2 and R_i for $i \geq 2$ do not intersect R_0.

We first prove the following useful result:

Proposition 2.3. *Let h be an arbitrary function which is continuous and piecewise differentiable on M and for which the Dirichlet integral $D[h; M]$ exists. A necessary and sufficent condition for a comparison funciton u to minmize the normalized Dirichlet integral $D[\hat{u}; M]$ is that, for all h,*

$$D[\hat{u}, h; M] = 0. \tag{2.8}$$

Proof. Let u be a comparison function that minimizes the normalized Dirichlet integral with $D[\hat{u}; M] = \mu$. Let h be as in the hypotheses of the theorem, so that $u + \lambda h$ is also a comparison function for every real constant λ. Then

$$0 \leq D[\hat{u} + \lambda h; M] - \mu = \lambda[2D[\hat{u}, h; M] + \lambda D[h; M]],$$

which implies that $D[\hat{u}, h; M] = 0$.

Conversely, if 2.8 holds for u and any comparison function h, let $h = v - u$, where v is an arbitrary comparison function. Then

$$D[\hat{v}; M] = D[\hat{u}; M] + D[h; M] \geq D[\hat{u}; M],$$

and $D[\hat{u}; M]$ minimizes the normalized Dirichlet integral. □

Now we can show that for a given covering $R_0, R_1, \ldots$, the comparison function u is uniquely determined if $D[\hat{u}; M] = \mu$ and $u - q$ vanishes at p_0. For if v is a function with these properties, then the function $h = v - u$ vanishes at p_0. By proposition 2.3, $D[\hat{u}, h; M] = 0$. Therefore $D[\hat{v}; M] = D[\hat{u}; M] + D[h; M]$. But $D[\hat{v}; M] = D[\hat{u}; M] = \mu$, which implies $D[h; M] = 0$, so $h = 0$ and $u = v$.

Proposition 2.4. *If the disks $R_1, R_2, \ldots$ are held fixed, then u is independent of the choice of disk R_0.*

Proof. We replace R_0 by a concentric disk R^* of radius $a_0 > a$ lying in the interior of R_1. Correspondingly, set

$$q^*(p_0) = q^*(x, y) = \frac{x}{x^2 + y^2} + \frac{x}{a_0^2}.$$

Also set $u^* = u - q^*$ in R^* and $u^* = u$ in $M - R^*$. By proposition 2.3, we only need show that for all admissible h,

$$D[\hat{u}, h; M] = 0 \tag{2.9}$$

implies

$$D[u^*, h; M] = 0$$

or equivalently

$$D[\hat{u} - u^*, h; M] = 0. \tag{2.10}$$

Assume equation (2.9). Then

$$u - u^* = \begin{cases} 0 & \text{in } M - R^* \\ q^* & \text{in } R^* - R_0 \\ q^* - q & \text{in } R_0, \end{cases}$$

and

$$D[\hat{u} - u^*, h; M] = D[q^* - q, h; R^*] + D[q, h; R^* - R_0].$$

As in step I of theorem 2.1, we apply Green's identity, this time to both summands on the right; observing that $q^* - q$ is harmonic in R^*, q is harmonic in $R^* - R_0$, $q_x dy - q_y dx$ vanishes on C_0, the boundary of R_0, and $q_x^* dy - q_y^* dx$ vanishes on C^*, the boundary of R^*, equation 2.10 follows, as desired. □

Proposition 2.5. *The minimum function u is independent of the choice of the remaining covering disks $R_1, R_2, \ldots$*

Proof. By the proof of proposition 2.4, if v is the extremum function associated with another covering with the same center p_0, then we can use the same disk R_0 for both coverings. Let $h = v - u$. By proposition 2.3, we have $D[\hat{u}, h; M] = D[\hat{v}, h; M] = 0$, from which we get

$$D[\hat{v}; M] = D[\hat{u}; M] + D[h; M] \qquad D[\hat{u}; M] = D[\hat{v}; M] + D[h; M] \tag{2.11}$$

and thus $D[h; M] = 0$, $h = 0$, and $v = u$. □

2.2.5 The Analytic Map

For our chosen point p_0, we have found a unique harmonic function u with our desired properties. From u we can obtain an analytic function by using the harmonic conjugate. Recall that for a harmonic function u, the harmonic conjugate v is given by

$$v(p) - c = \int_L (u_x dy - u_y dx) \tag{2.12}$$

where c is an arbitrary real constant, and L is a rectifiable curve on $\dot{M}$ from a fixed point p_1 to a variable point p, and integration is with respect to the appropriate local uniformizing parameters.

Proposition 2.6. *If M is simply connected, then the harmonic conjugate $v(p)$ of the minimum function u is single-valued.*

Proof. Let $p_1 \in \dot{R}_0$ and L be closed. Our proposition will follow if we show that the integral (2.12) over L is zero. Since M is simply connected, our only concern is at p_0. L is homotopic to a curve in R_0 which loops around p_0, so it is sufficient to consider L to be such a curve in R_0. Thus, all we need to show is that u has a single-valued harmonic conjugate in R_0. We write $u = (u - q) + q$. Since $u - q$ is harmonic throughout R_0, a simply connected disk, it has a single-valued harmonic conjugate there. Also, we chose q to be the real part of an anlytic function, so in $\dot{R}_0$ it has the single valued harmonic conjugate $-y/(x^2 + y^2) + y/a^2$. Then u has a single valued harmonic conjugate. □

Setting $f(p) = u(p) + iv(p)$, we have obtained a function regular and single-valued on the simply connected surface $\dot{R}$ and having a simple pole with residue 1 at p_0.

2.3 Bijectivity of the Analytic Map

We now study the mapping $w = f(p)$ from the simply connected region M into the w-sphere. We want to show that f maps M bijectively onto its image. The mapping will be conformal since f is analytic, and the image will also be simply connected.

Our procedure is motivated by our previous physical considerations of the dipole in a simply connected region of the plane. Recall that our insight was gained from tracing the flow lines $v = c$ on our region. Our picture showed each such line dividing the region into two connected domains, with $v > c$ and $v < c$. Our first step is to prove that this is true on our surface. From this fact we can prove that our map is locally injective. Then we will prove, as we expect from our picture, that there is only one curve $v = c$ that is not closed, but extends out to the boundary; it will be our endles curve. The last step will be to consider the images of our Riemann surface under the map with the aid of our constant v lines. The result will be three cases, which will be our three domains: the unit disk, the plane, and the sphere.

2.3.1 Local Injectivity

At the point $p = p_0$, the function $1/w$ has a simple zero. Then f maps a neighborhood of p_0 conformally onto a neighborhood of $w = \infty$. In the neighborhood M_1 of p_0, we have

$$u + iv = w = \frac{1}{z} + c_1 z + \ldots, \tag{2.13}$$

$$u = \frac{x}{x^2 + y^2} + a_1 x + \ldots, \tag{2.14}$$

$$v = -\frac{y}{x^2 + y^2} + b_1 y + \ldots. \tag{2.15}$$

Let c be any real constant. The equation $v = c$ determines a line in the w-plane parallel to the u-axis, which we think of as running from $u = \infty$ to $u = -\infty$. For sufficiently large values of $|c|$, these parallels are images under f (which is locally conformal at p_0) of small simple closed curves touching the x-axis at $z = 0$ an approximating the circles given by $c(x^2 + y^2) = -y$. We wish to trace the curves on M that are mapped by f onto the various lines $v = c$ for arbitrary c.

Let us fix c. If $1/u$ varies over a small interval about 0, then

$$z = w^{-1} + c_1 w^{-3} + \ldots = (u + ic)^{-1} + \ldots = u^{-1} - icu^{-2} + \ldots$$

varies over a small arc touching the x-axis at the origin, since the sign of $\mathrm{Im}(z)$ is always constant, either negative or positive. this arc divides a sufficiently small disk on M with center p_0 into only two domains. Let $\mathfrak{H}$ denote the lower of these two domains. Then $v > c$ in the lower domain, and $v < c$ in the upper domain, since as c increases, $\mathrm{Im}(z) = -cu^{-2}$ decreases. We have the following lemma.

Lemma 2.4. *All the points of $\dot{M}$ at which $v(n) > c$ form a single domain.*

Proof. We must show that every such point n can be joined to a point n_1 of $\mathfrak{H}$ by a curve on $\dot{M}$ at all of whose points we have $v > c$. Our proof is by contradiction. Assume there is a point n_2 on $\dot{M}$ with $v(n_2) > c$ which fails to have this property. Let $\mathfrak{G}_k, (k = 1, 2)$ be the set of all points of $\dot{M}$ which can be joined to n_k by curves for which $v > c$. These two sets form disjoint domains with p_0 a boundary point of $\mathfrak{G}_1$ but not of $\mathfrak{G}_2$. Let $g(v)$ be a real valued function that is continuously differentiable for all finite $v \geq 0$, is bounded, has bounded derivative, is positive for positive v, and satisfies $g(0) = 0, g'(0) = 0$. One such function is

$$g(v) = \frac{v^2}{1 + v^2} = 1 - \frac{1}{1 + v^2}.$$

Let $k(u)$ be a real valued function that is bounded, is continuously differentiable for all real u, and has a derivative that is bounded and positive for all u. One such function is

$$k(u) = \arctan u, \text{ for } -\frac{\pi}{2} < \arctan u < \frac{\pi}{2}$$

Let $h(p) = k(u)g(v - c) = kg$ for all p in $\mathfrak{G}_2$ and $h(p) = 0$ in $M - \mathfrak{G}_2$. then $h(p)$ is continuously differentiable in terms of local parameters throughout M. In $\mathfrak{G}_2$,

$$h_x = k'gu_x + kg'v_x = k'gu_x - kg'u_y \tag{2.16}$$

$$h_y = k'gu_y + kg'v_x = k'gu_y + kg'u_x \tag{2.17}$$

$$u_x h_x + u_y h_y = k'g(u_x^2 + u_y^2) \tag{2.18}$$

$$h_x^2 + h_y^2 = [(k'g)^2 + (kg')^2](u_x^2 + u_y^2), \tag{2.19}$$

and in $M - \mathfrak{G}_2$, both h_x and h_y vanish. Then since we have $(k'g)^2 + (kg')^2$ is bounded and $D[\hat{u}; M]$ coverges, we find that $D[h; M]$ exists. Since u is independent of the choice of M_0, we assume that M_0 lies entirely in $M - \mathfrak{G}_2$. This is possible because p_0 is not in $\mathfrak{G}_2$ or its boundary. Then

$$D[\hat{u}, h; M] = D[u, h; \mathfrak{G}_2] = \iint_{\mathfrak{G}_2} k'(u)g(v-c)(u_x^2 + u_y^2)dxdy \tag{2.20}$$

where

$$u_x^2 + u_y^2 = |\frac{df}{dz}|^2. \tag{2.21}$$

It follows that the integrand > 0 except possibly for isolated points of $\mathfrak{G}_2$. This implies that $D[\hat{u}, h; M] > 0$, which contradicts proposition 2.3. □

To get that $v < c$ defines a single domain, we replace $g(v-c)$ with $g(c-v)$ in the above proof. After proving this lemma, we can now show that f is locally injective.

Theorem 2.2. *The mapping $w = f(p)$ is conformal and locally injective at all points of M relative to local parameters.*

Proof. We already saw that f is locally injective at p_0. Now we show that on $\dot{M}$ the derivative of $f(p)$ with respect to the local parameter is not zero, and this will be sufficient to prove the theorem. Assume there exists a point p^* at which f vanishes to order $l > 1$ in terms of the local parameter t. Then f maps a neighborhood of $t = 0$ onto a neighborhood of $w^* = f(p^*)$ covered l times. The inverse map of the line $v = v^*$ through $w^* = u^* + iv^*$ gives l curves on M near p^* that intersect at p^* and form $2l$ sections with angle π/l. Going around p^*, the sections alternate having $v > v^*$ or $v < v^*$. Now choose n_1, n_3 in disjoint sectors with $v > v^*$, and n_2, n_4 in disjoint sectors with $v < v^*$. By part I, we can find a curve L_1 that joins n_1 to n_3, with $L_1 \subset \dot{M}$ and $v > v^*$ on all of L_1, and we can find a curve L_2 that joins n_2 to n_4, with $L_2 \subset \dot{M}$ and $v < v^*$ on all of L_2. Join each of n_1, n_2, n_3, n_4 to p^* by curves that remain in their respective sectors. We get two closed curves L_1, L_2 which cross only at p^*, and the index, or intersection number, of L_1 and L_2 is ± 1. But by the simple-connectedness of M, we can contract L_2 to a point outside of L_1, so that the index of L_1 and L_2 is 0. This contradicts the fact that the index of two closed curves on a Riemann surface is invariant under continuous deformations of the curves throughout which the number of common points of the two curves remains finite. Thus our theorem is proved. □

2.3.2 The Endless Curve

Let us now look at curves L extending from p_0 on M. Let

$$z = \zeta(w) = w^{-1} + c_1 w^{-3} + \cdots$$

be the function element (local representation of z as a function of w) at $w = \infty$ obtained by inverting $w = f(p)$ at $p = p_0$. This function element can be continued analytically along the line $v = c$ on the w-sphere, for fixed c. We can have u increase from $-\infty$ or decrease from ∞. There are two cases:

1. Analytic continuation is possible on the whole line $v = c$, and we come back to the initial function element $\zeta(w)$ at ∞, in which case there is a corresponding curve L in R which is mapped by f onto C. Since f is single-valued and infinite only at $p = p_0$, L is a simple closed curve;

2. Analytic continuation is only possible from $-\infty$ up to a and from ∞ down to b; then we have rays $u < a$ and $u > b$ which are images of the curves L_1 and L_2 on R, respectively. Again by the single-valuedness of f, L_1 and L_2 have no double point and form a simple curve on R. If they had a common point other than p_0, then analytic continuation back to ∞ would be possible; so they have no common point.

In the second case, we wish to know what happens to p on L_1 as u approaches a. We claim that for a compact subset N of R, there exists a δ such that for $a - \delta < u < a$, p is outside of N. Suppose such a δ did not exist. Then there would exist a monotonically increasing sequence u_n converging to a such that the images p'_n on N converge to a point $p' \neq p_0$ in N. Since f is continuous, we have $f(p') = a + ci$, and we could continue $\zeta(w)$ up to an including a. This is a contradiction, so our claim is proved. As a consequence, we see that as u approaches a, the point p on L_1 leaves every compact set B_n; the same holds for L_2. Thus we call the curve $L = L_1 + L_2$ an *endless curve*.

Now we prove the following theorem.

Theorem 2.3. *An endless curve occurs for at most one value of* c.

Proof. Again, we prove by contradiction. Let $v = c, v = c^*$ define two endless curves L, L^*, with L_1 the subarc of L corresponding to u increasing from $-\infty$, L_2 the subarc of L corresponding to u decreasing from ∞, and similarly for subarcs L_1^*, L_2^* for L^*. L and L^* have only p_0 in common, since f is single-valued. They touch at the x-axis without crossing, in terms of the local coordinate $z = x + iy$ at p_0. Let $\mathfrak{D}$ be a disk centered at p_0 small enough such that $L_1, L_1^*; L_2, L_2^*$ determine two sectors $\mathfrak{S}_1, \mathfrak{S}_2$. We claim that *every interior point* $n_1 \in \mathfrak{S}_1$ *can be joined to* $n_2 \in \mathfrak{S}_2$ *by a curve for which* $c < v < c^*$. For $-m$ sufficiently large and positive, the line $u = m$ in the w-plane has image C, approximately a small circle, contained in $\mathfrak{D}$. C cuts off a sector of $\mathfrak{S}_1$, and does not intersect $\mathfrak{S}_2$. Let n_1 be a point lying in $\mathfrak{S}_1$ but outside of C. Consider the set of points on $\dot{M}$ such that $u > m$ and $c < v < c^*$; this set may form disjoint domains, so we let $\mathfrak{G}$ be such a domain containing n_1. Assume for contradiction that $n_2 \notin \mathfrak{G}$. Then p_0 is not a boundary point of $\mathfrak{G}$, and by proposition 2.4 we can assume that M_0 lies in $M - \mathfrak{G}$.

Now let g be as in theorem 2.2, e.g., $g = v^2/(1+v^2)$. Define $h(p) = g(u-m)g(v-c)g(c^*-v)$ on $\mathfrak{G}$, and $h(p) = 0$ on $M - \mathfrak{G}$. Put

$$g'(u-m)g(v-c)g(c^*-v) = g_1$$
$$g(u-m)(g'(v-c)g(c^*-v) - g(v-c)g'(c^*-v)) = g_2.$$

In $\mathfrak{G}$, we have:

$$h_x = g_1 u_x - g_2 u_y, h_y = g_1 u_y + g_2 u_x$$
$$u_x h_x + u_y h_y = g_1(u_x^2 + u_y^2),\ h_x^2 + h_y^2 = (g_1^2 + g_2^2)(u_x^2 + u_y^2).$$

Then $D[h; M]$ exists, and $D[\hat{u}, h; M] > 0$, which contradicts proposition 2.3. So n_2 lies in $\mathfrak{G}$.

Now we join n_1 and n_2 by a curve lying in $\mathfrak{G}$, i.e., for which $c < v < c^*$; we close this curve by tracing a curve lying in $\mathfrak{S}_1$ from n_1 to p_0, and then a curve lying in $\mathfrak{S}_2$ from p_0 to n_2. We obtain a closed curve B. Let A be the curve $L_1 + L_2^*$. Then the index of A and B is ± 1. Since B is closed and A extends to infinity in both directions, the index must be invariant under continuous deformations of B that keep the intersection points finite. But again by the simple connnectedness of M, the curve B can be contracted to a point outside A, so that the index of A and B is 0. This contradition shows that there are no two endless curves L, L^*. □

Let L be and endless curve on which $v = c$. Let L_1 with $u < a$ and L_2 with $u > b$ be two subarcs of L which meet at p_0. Then we can show that $a \leq b$. For, suppose $b < a$, and let $d = (a+b)/2$. For $c_1 < c$, consider the strip $c_1 \leq v < c$. Since L is the only endless curve, $z = \zeta(w)$ admits unique analytic continuation to all points of the strip. Adjoining the rays corresponding to L_1 and L_2, we have that analytic continuation along these rays yields the same function elements at d. But then continuation of L_1 beyond $u = a$ is possible, which is a contradiction.

2.3.3 The 3 Cases

Now we have 3 cases:

1. There is no endless curve.
2. There exists one endless curve $v = c$ with $a = b$.
3. There exists one endless curve $v = c$ with $a < b$.

To each case corresponds an image W of the map:

1. The full w-sphere. This is the elliptic case.
2. The sphere punctured at $w_1 = a + ci$. This is the parabolic case.
3. The sphere slit along $a \leq u \leq b, v = c$ from $w_1 = a + ci$ to $w_2 = b + ci$, with the edges omitted. This is the hyperbolic case.

Theorem 2.4. *The function $w = f(p)$ effects a biholomorphic mapping of the simply connected Riemann surface M onto W.*

Proof. We have already shown the map to be analytic locally injective. Here we prove that the map is globally bijective. Let $n_1 \in M$ and $c = \text{Im}(f(n_1)) = v(n_1)$. Consider the curve L on M arising from analytic continuation of $z = \zeta(w)$ at $w = \infty$ along $v = c$. The curve L is either closed or endless, with $v > c$ on one side and $v < c$ on the other.

We claim that n_1 *is on* L. Suppose n_1 is not on L. Then we can find n_2, n_3 sufficiently close to n_1 such that $v(n_2) > c$ and $v(n_3) < c$. The points n_2, n_3 can be joined by a curve not meeting L. Then we can obtain a closed curve B through p_0 that crosses L only at p_0. Again, considering the index of L and B when B is contracted to a point outside L, we get a contradiction. We can conclude that every point of M lies on a curve L. Conversely, for a given c, there is a unique correspondence between the curves L and the lines $v = c$ on W via the mapping $w = f(p)$. Then f covers W. Further, if $f(p) = f(n_1)$, then p must also lie on L. Since every point on L is uniquely determined by the value of u, we find that $p = n_1$, and f is one to one. □

Now in the elliptic case, W is the full sphere and is compact. So M is compact and simply connected, so it must have genus 0. In the parabolic case, the transformation $s = (w - w_1)^{-1}$ maps the punctured w-sphere conformally onto the full s-plane. The composition $s(f(p))$ maps the Riemann surface M conformally onto the full s-plane. In the hyperbolic case, let

$$s = \frac{\sqrt{w - w_1} - \sqrt{w - w_2}}{\sqrt{w - w_1} + \sqrt{w - w_2}}$$

with the branch satisfying $0 < s < 1$ for positive values of $w - w_2$ on w-sphere slit from w_1 to w_2. Then s maps W conformally onto the interior of the unit circle in the s-plane. The composition $s(f(p))$ maps the Riemann surface M conformally onto the interior of the unit circle in the s-plane.

Hence we have completed the first proof of the uniformization theorem for simply connected Riemann surfaces.

2.4 Arbitrary Riemann Surfaces

We have shown that every simply connected Riemann surface is conformally equivalent to the sphere, the plane, or the unit disk. Now we use the universal covering surface to describe any arbitrary Riemann surface in terms of these three spaces. It is known that every Riemann surface has a unique universal covering surface, up to conformal equivalence. The universal covering surface is defined as being simply connected. Then if M is a Riemann surface, the universal covering surface $\widetilde{M}$ is conformally equivalent to the sphere, the plane, or the unit disk. We a free to assume that $\widetilde{M}$ is one of these spaces.

Let $\pi : \widetilde{M} \to M$ be the projection map. Recall that a homeomorphism $\phi : \widetilde{M} \to \widetilde{M}$ is a cover transformation if $\pi \circ \phi = \pi$. It is known that every cover transformation is a conformal homeomorphism, and every cover transformation that is not the identity has no fixed points.

For the sphere, the plane, and the disk, all conformal self-mappings are given by the projective transformations (also called fractional linear or Möbius transformations, which have the form

$$\phi(z) = \frac{az+b}{cz+d},\ ad - bc \neq 0. \tag{2.22}$$

Every such mapping has at least one fixed point on the sphere. Therefore, if $\widetilde{M}$ is the sphere, the only cover transformation is the identity. If $\widetilde{M}$ is the plane, the cover transformations must fix the point at infinity, so they are given by parallel translations $\phi(z) = z + b$. If $\widetilde{M}$ is the unit disk, then for cover transformations that are not the identity, the fixed points must lie on the unit circle. Then any cover transformation ϕ has the form $\phi(z) = (az + b)/(\bar{b}z + \bar{a})$.

The group of cover transformations is isomorphic to the fundamental group $\pi_1(M)$. If $\widetilde{M}$ is the sphere, then $\pi_1(M) = 1$. Since the sphere is compact, so is its projection, and M is conformally a sphere.

In the other two cases, we make use of the fact that the group Γ of cover transformations is properly discontinuous on $\widetilde{M}$. When $\widetilde{M}$ is the plane, the group of cover transformations is a properly discontinuous group of parallel translations. there are only three types of such groups: (1) the identity, (2) the infinite cyclic group generated by $\phi(z) = z + b,\ b \neq 0$, (3) the abelian group generated by $\phi_1(z) = z + b_1$ and $\phi_2 = z + b_2$, with nonreal ratio b_2/b_1. By identifying points of the plane that correspond under the transformations in Γ, we get back M. In each of the three cases above, M is (1) the plane, (2) an infinite cylinder, conformally equivalent to the punctured plane, and (3) a torus obtained by identifying opposite sides of a parallelogram.

For all other Riemann surfaces not listed above, $\widetilde{M}$ is the unit disk, and Γ is a properly discontinuous group of fixed point free linear transformations mapping the disk onto itself. Conversely, given such a group Γ, we can obtain a Riemann surface by identifying the points of the disk that are equivalent under this group.

Via these results of uniformization, the study of Riemann surfaces can be reduced to the study of the sphere, the plane, and the disk, along with the groups of fixed point free self-mappings of these spaces that are projective transformations.

3 The Second Proof

3.1 Preliminaries

In our first proof of the uniformization theorem, we described the Riemann surface as a collection of linked disks with conformal structure, and our main construction used harmonic functions. In our second proof, we will use the triangulation of a Riemann surface, and we will deal with analytic functions in constructing our map. A surface can be defined combinatorially in terms of triangulation, and then the conformal structure added afterwards to make it a Riemann surface (see Sansone [8] for definitions). A surface is *closed*, or compact, if the number of triangles in the triangulation is finite, and *open* if it is countably infinite. Covering surfaces and simple connectedness can be defined in these terms. For consistency, we note that the triangulability of a Riemann surface can be derived from the countability axiom, which we used in section 2 (see Ahlfors and Sario [4]). Another observation that we need is the following: it is possible to subdivide triangles of a general RS such that every one is included in a local coordinate chart.

The statement of the uniformization theorem to be proved here is the same as in section 2, and the methods here are classical as well. The mapping function will be constructed using a convergent sequence of functions. We saw in the first proof that the covering of disks was countable and ordered, and we would expect that the same is possible for our triangulation. The precise ordering we need is given by Van der Waerden's lemma (Sansone [8]).

Lemma 3.1. *If M is a simply connected open surface, then its traingles can be enumerated in such a way that every triangle Δ_{n+1} has one side or two sides in common with the sum of the preceding triangles $E_n = \Delta_1 + \ldots + \Delta_n$ but not a side and the opposite vertex.*

As in our first proof of uniformization, an inductive method is suggested to us; in this case, we will use continuation and the Schwarz reflection principle for analytic functions. A key tool for getting analytic maps will be the Riemann mapping theorem 1.1. We note that the Schwarz reflection principle requires that the boundary being reflected across is an analytic arc, and that the analytic function being reflected extends continuously to the boundary. In the following proof, the arcs being reflected across are all circular arcs, and the function being reflected is derived from the Riemann mapping theorem. The regions being mapped will all be bounded by analytic arcs or circular arcs, in which case it is known that the mapping extends continuously to the boundary. In this section, where the Riemann mapping theorem is invoked, this extension is implied.

We will need one futher statement concerning normal families. Recall that a family of functions is normal in a region R if every sequence of functions in the family contains a subsequence that converges uniformly on every compact subset of R. We have the following lemma which can be derived from the Koebe Distortion Theorem:

Lemma 3.2. *If the family of univalent and holomorphic functions in a region R is such that the values of the functions as well as those of their derivatives are bounded at a given point of R, then the family is a normal family.*

Now we provide our second proof of the uniformization theorem.

3.2 Proof of the Uniformization Theorem

Let M be a simply connected Riemann surface with local uniformizing parameters t. We triangulate M such that every triangle lies in a neighborhood having a local uniformizing parameter. In the t-plane, the triangles look like simply connected regions bounded by three analytic arcs.

3.2.1 Case 1: Open

We first consider the case that $\widetilde{M}$ is open, and we assume that the triangles $\Delta_1, \Delta_2, \ldots$ of its trianguation are ordered according to Van der Waerden's lemma.

Let Δ_1 be included in a neighborhood U in which the local parameter t is defined. Δ_1 corresponds to a simply connected region Δ' in the t-plane, bounded by analytic arcs. Then by the Riemann Mapping Theorem, Δ' can be mapped conformally onto the interior of a circle in the s plane such that there is a one to one correspondence between the boundaries.

Now we proceed by induction. Assume $E_n = \Delta_1 + \cdots + \Delta_n$ is mapped conformally onto an open disk M_n in the s-plane, with a one to one correspondence between boundaries. Let $s = \phi_n(\mathfrak{p})$ denote the mapping function. We wish to obtain such a map for E_{n+1}. The region E_{n+1} consists of E_n plus a triangle $\Delta = \Delta_{n+1}$ which shares one or two sides with E_n. Δ lies in a neighborhood U in $\widetilde{M}$ which is mapped injectively onto a neighborhood U' in the t-plane by the local parameter t. Δ is mapped by t onto Δ' in U'. A common side a of E_n and Δ is mapped by ϕ_n continuously onto an arc a_1 of the boundary of R_n, and by t onto an arc a' of U'. Let V be the region of E_n that lies in U of which a forms part of the boundary. The image V_1 of V in the s-plane lies in R_n and has a_1 as part of its boundary. Let V' in U' be the image of V in the t-plane. Via the local parameter, the function $s = \phi_n(t)$ maps V' to V_1, where $\phi_n(t)$ is the restriction of $\phi_n(p)$ to U. Now draw in R_n a circular arc a_2 with endpoints correpsonding with those of a_1. At its endpoints, the arc a_2 should have an angle $\pi/2^r$ relative to a_1 small enough so that a_2 lies in V_1; such an arc is possible for large enough r. Denote the 2-gon enclosed by a_1 and a_2 by B_1. The image of B_1 in $\widetilde{M}$ is B, which lies in V, and its image B' in the t-plane lies in V'.

After this somewhat tedious preliminary setup, we briefly sketch our situation and the steps that follow. We do not yet know how to map E_{n+1} to a domain in the plane, but we do have a maps on E_n and Δ separately. Our key is the domain B, conformally equivalent to B_1 and B', which is common to our two maps t and ϕ_n, and which we have nicely set up to take advantage of the Schwarz reflection principle. So roughly, the idea is to map the two parts of E_{n+1} separately to the same disk, and then to make sure that the maps agree on B. Most of the work is in getting the maps from $\Delta' + B'$ in the t-plane, and from R_n in the s-plane.

We claim that there are holomorphic functions

$$s^* = g(s), \qquad\qquad s^* = G(t) \tag{3.1}$$

such that g maps the disk R_n onto a region $B^* + H^*$ and G maps $\Delta' + B'$ onto a region $\Delta^* + B^*$, with the following four properties:

1. $\Delta^* + B^* + H^*$ constitutes an open disk R_{n+1};
2. $s^* = g(s)$ maps B_1 onto B^* and $H_1 = R_n = B_1$ onto H^*;
3. $s^* = G(t)$ maps B' onto B^* and Δ' onto Δ^*;
4. in B' holds the relation
$$G(t) = g(\phi_n(t)). \tag{3.2}$$

We start with the simply connected region $\Delta' + B'$ in the t-plane. By the Riemann mapping theorem, it can be mapped by a function $t_1 = A_1(t)$ onto a disk $\Delta^{''} + B_1^{''}$ with the center of the disk in the interior of $\Delta^{''}$. Then the mapping given by

$$t_1 = A_1(\phi_n^{-1}(s)) = g_1(s)$$

relates B_1 conformally to $B_1^{''}$, where $t = \phi_n^{-1}(s)$ from V_1 to V' is the inverse map of $s = \phi_n(t)$. By the Schwarz reflection principle, we can extend this mapping to B_2, the reflected image of B_1 with respect to a_2. The image of B_2 under $(g_1(s)$ is $B_2^{''}$, which is the reflection of $B_1^{''}$ with repsect to $a^{''}$, the image of a_2. Then the region $\Delta^{''} + B_1'' + B_2^{''}$ is simply connected and can be mapped onto a circular disk $\Delta''' + B_1''' + B_2'''$ such that the center of the disk lies in $\Delta^{'''}$. Then the mapping

$$t_2 = A_2(g_1(s))$$

maps the 2-gon $B_1 + B_2$ with angle $\pi/2^{r-1}$ and sides a_1 and a_3 to $B_1''' + B_2^{'''}$. We again apply the reflection principle to extend the mapping to B_3, the reflection of $B_1 + B_2$ with respect to a_3. The image of B_3 is $B_3^{'''}$, which is the reflection of $B_1''' + B_2^{'''}$ with repsect to $a^{'''}$ (the image of a_3). The resulting region $\Delta^{'''} + B_1''' + B_2^{'''} + B_3^{'''}$ is again simply connected and can be mapped to a disk. This process can be continued; after r steps, R_n is wholly covered by two-gons, and we have two mappings

$$s^* = A_{r+1}A_r \dots A_2A_1(t) = G(t) \tag{3.3}$$

$$s^* = A_{r+1}A_r \dots A_2A_1\phi_n^{-1}(s) = g(s). \tag{3.4}$$

$G(t)$ maps Δ' and B' onto Δ^* and $B^* = B_1^{(r+2)}$; g(s) maps B_1 and $H_1 = B_2 + \cdots + B_{r+1}$ onto $B^* = B_1^{(r+1)}$ and $H^* = B_2^{(r+2)} + \cdots + B_{r+1}^{(r+2)}$; $\Delta^* + B^* + H^*$ is an open disk. This verifies properties 1, 2, and 3. Property 4 follows from

$$g(\phi_n(t)) = A_{r+1}A_r \dots A_2A_1\phi_n^{-1}(\phi_n(t)) = A_{r+1}A_r \dots A_2A_1(t) = G(t),$$

and our claim is proved.

Remark 3.1. The proof of our claim started with $\Delta + B'$ and then obtained maps from the s-plane that incorporated larger and larger regions of R_n. With some modifications, we could have started with R_n and obtained maps from the t-plane incorporating larger and larger regions of Δ via reflection.

Now we pull back the maps $g(s)$ and $G(t)$ in the t- and s-planes to our original region E_{n+1} on the Riemann surface. The function

$$s^* = g(s) = g(\phi_n(p)) = g^*(p)$$

maps E_n onto $B^* + H^*$ and the function

$$s^* = G(t) = g^{**}(p)$$

maps $\Delta + B$ onto $\Delta^* + B^*$. If p is in B, then $g^*(p)$ and $g^{**}(p)$ are defined and we have

$$g^*(p) = g(s) = g(\phi_n(t)) = G(t) = g^{**}(p).$$

Now $g^*(p)$ is defined throughout E_n, $g^{**}(p)$ throughout $\Delta + B$. Hence they are analytic continuations of each other, and they define a function

$$s^* = \phi_{n+1}(p)$$

which maps $E_{n+1} = E_n + \Delta$ onto the disk R_{n+1}. We have proved the following statement:

The interior of every of the infinitely many regions $E_n = \Delta_1 + \cdots + \Delta_n$ of an open covering surface $\widetilde{M}$ can be mapped one to one and conformally onto an open disk R_n. The boudnary of E_n corresponds one-to-one and continuously to the circumference of R_n.

The mapping function for E_n is denoted $\phi_n(p)$. Selecting a point $\mathfrak{o}$ in the interior of Δ_1, we can normalize these functions by setting

$$\phi_n(\mathfrak{o}) = 0, \qquad \phi_n'(\mathfrak{o}) = 1 \tag{3.5}$$

for the functions and their derivatives with repsect to local parameters.

Now we form the functions

$$\phi_{1,n}(s) = \phi_n(\phi_1^{-1}(s))$$

for $n = 1, 2, \ldots$, which are holomorphic in R_1, univalent, and normalized at $s = 0$. Thus by lemma 3.2, they constitute a normal family. We select a subsequence which converges on the interior of R_1 to a univalent function. We may do the same for the sequence

$$\phi_1(p), \phi_2(p), \ldots$$

which contains a subsequence

$$\phi_1^1(p), \phi_2^1(p), \ldots$$

which converges in the interior of E_1 to a univalent holomorphic function $\phi_0(p)$. Since we may suppose that R_2 again is in the s-plane, we can, as above, construct the functions

$$\phi_{2,n}(s) = \phi_n(\phi_2^{-1}(s))$$

and get the sequence

$$\phi_1^2(p), \phi_2^2(p), \ldots$$

that converges throughout E_2 to a function whose restriction to E_1 is ϕ_0. We call this function ϕ_0 as well.

Repeating this process and applying the diagonal principle, we get a sequence

$$\phi_1^1(p), \phi_2^2(p), \ldots$$

where $\phi_k^k(p)$ is defined throughout E_n if $k \geq n$ and converges to ϕ_0 there. Since E_n exhaust the surface $\widetilde{M}$, we see that $\phi_0(p)$ is univalent on $\widetilde{M}$ and maps $\widetilde{M}$ onto a region R in the s-plane. It is easy to see that R is simply connected. Since $\widetilde{M}$ is open, R cannot be the extended plane; therefore, it is either the whole plane, or it can by mapped by the Riemann Mapping theorem to the unit disk.

3.2.2 Case 2: Closed

Now we consider the case that $\widetilde{M}$ is closed. Then $\widetilde{M}$ consists of a finite number of triangles making up E_n, and a closing triangle Δ which has three sides in common with E_n. Puncture Δ by an interior point q. The remaining part is a simply connected open surface $\widetilde{M}_0$ which can be mapped onto either a plane or an open disk; denote this map by $s = \phi(p)$. We will show that only the former case can occur.

Let $\mathfrak{o}$ be a point in $E = E_n$. Let E' be the image of E and Δ' of $\Delta\backslash\{q\}$ under ϕ. Assume $s = 0$ corresponds $\mathfrak{o}$. Let R' be a disk about $s = 0$ contained in E'. Then Δ' is outside the disk, and so the function $w = 1/s$ maps Δ' to a bounded region of the w-plane. Considering $w = 1/\phi(p)$ as a function on $\widetilde{M}\backslash\{q\}$, w is bounded and analytic in a neighborhood near q, so w has a removable singularity at q. Let q'' denote the image of q in the w-plane.

We argue by contradiction. Suppose that the image of the punctured surface is the interior of a disk R in the s-plane, hence the exterior of a disk $R^{''}$ in the w-plane. A sequence of points $p_1^{''}, p_1^{''}, \ldots$ outside $R^{''}$ and having an accumulation point on the circumference corresponds to a sequence $p_1, p_2, \ldots$ on $\widetilde{M}$ that also has an accumulation point, since $\widetilde{M}$ is closed. Then this accumulation point must be q, for any other accumulation point on $\widetilde{M}$ would be mapped to the exterior of R'' and not a point on the circumference. Thus $q = \lim_{n\to\infty} p_n$, and $q^{''} = \lim_{n\to\infty} p_n^{''}$. But the accumulation point of $p_1^{''}, p_1^{''}, \ldots$ can be selected arbitrarily on the circumference of $R^{''}$. Thus to q there must correspond infinitely many points $q^{''}$, and we have a contradiction.

There is no contradiction if the radius of $R^{''}$ is zero and $q^{''}$ is the origin. Then in the s-plane, the radius of R is infinite, and thus $\widetilde{M}$ corresponds to the extended plane. The proof of the our theorem is complete. □

4 Uniformization with Projective Structures

In the time since the uniformization theorem was first proved by the methods of sections 2 and 3, newer tools have been developed and applied to the study of Riemann surfaces. In this section, we take a more algebraic approach to uniformization using sheaf cohomology, following Gunning [6], [7]. An advantage of this approach is the possibility of generalizing it to higher dimensions. We present Gunning's argument leading up to a geometric realization derived from special structures on a Riemann surface, termed projective structures. The final step of deriving the uniformization theorem from the geometric realization was not demonstrated by Gunning; we conclude with a brief discussion of approaches to this problem.

4.1 Projective Structures

Our definition of a Riemann surface will be essentially the same as we have been using in the previous sections. We present it here with emphasis on the formalism that will be specialized to the affine and projective cases. A coordinate covering $\{U_\alpha, z_\alpha\}$ of a 2-dimensional manifold M will be called a *complex analytic coordinate covering* if all the coordinate transition functions are holomorphic functions. Two complex analytic coordinate coverings are *equivalent* if their union is also a complex analytic coordinate covering. We will call an equivalence class of complex analytic coordinate coverings a *complex (analytic) structure* on M. Then a *Riemann surface* is a surface M with a fixed complex structure. Note that the only property of holomorphic functions needed for these definitions is that holomorphic fucntions are closed under composition when composition is defined. This is called the *pseudogroup property*.

By the above discussion, we can define a structure on a surface whenever we have a pseudogroup. We will be interested in the affine and projective (pseudo)groups, which are subsets of the set of complex analytic local homeomorphisms. The uniformization theorem tells us that the universal covering surface of any Riemann surface is a sphere, plane, or unit disk, and the cover transformations are given by certain projective transformations, a consequence of the fact that all conformal self-maps of the sphere and disk are projective, and those of the plane are affine. We let $\mathcal{A}$ denote the group the complex affine mappings, and $\mathcal{P}$ deonte the complex projective mappings. A function $f \in \mathcal{A}$ has the form

$$f(z) = az + b,$$

for some complex constants $a \neq 0, b$. A function $f \in \mathcal{P}$ has the form

$$f(z) = \frac{az + b}{cz + d}$$

for constants $a, b, c, d, ad - bc \neq 0$. This last condition implies that $f'(z) \neq 0$ We note for later that any uniformization of a Riemann surface provides a coordinate covering of the surface such that the local analytic coordinates are related by projective transformations.

It will be convenient to describe affine and projective transformations by certain differential operators. Let $f : U \to V$ be a complex analytic local homeomorphism, where U, V are open subdomains of $\mathbb{C}$. We introduce 2 differential operators θ_1, θ_2:

$$\theta_1 f(z) = \frac{f''(z)}{f'(z)} \tag{4.1}$$

$$\theta_2 f(z) = \frac{2f'(z)f'''(z) - 3f''(z)^2}{2f'(z)^2} \tag{4.2}$$

Since f is a local homeomorphism, $f'(z) \neq 0$ for all $z \in U$, and thus $\theta_\nu f$ are holomorphic throughout U for $\nu = 1, 2$. We call θ_2 the *Schwarzian derivative*. Note that if $g : V \to W$ is another complex analytic local homeomorphism, then so is the composition $h = g \circ f : U \to W$. A straightforward calculation shows that

$$\theta_\nu h(z) = \theta_\nu g(w) \cdot f'(z)^\nu + \theta_\nu f(z) \qquad \text{for } \nu = 1, 2, \tag{4.3}$$

with $w = f(z)$.

Now let $\mathcal{F}_\nu$ be the family of all complex analytic local homeomorphisms such that $\theta_\nu f(x) = 0$ at all points z where f is defined. Then it is clear from equation (4.3) that $\mathcal{F}_\nu$ has the pseudogroup property. As expected, $\mathcal{F}_1$ consists of the *complex affine mappings*, and $\mathcal{F}_2$ consists of the *complex projective mappings* (or linear fractional or Möbius transformations).

We define affine and projective structures in the obvious way. Let $\{U_\alpha, z_\alpha\}$ be a coordinate covering of a manifold M with transition functions $z_\alpha = f_{\alpha\beta}(z_\beta)$. In analogy to the complex analytic case, the covering will be called an $\mathcal{F}_\nu$ *coordinate covering* if the transition fuctions $f_{\alpha\beta}$ belong to $\mathcal{F}_\nu$. Two coordinate coverings are *equivalent* if their union is also an $\mathcal{F}_\nu$ coordinate covering. Then an equivalence class of $\mathcal{F}_\nu$ coordinate coverings is an $\mathcal{F}_\nu$ *structure* on M. The adjectives affine and projective will be used for the appropriate structures.

Note that an affine coordinate covering is also a projective coordinate covering, since $\mathcal{A} \subset \mathcal{P}$. This also means that two equivalent affine coordinate coverings are equivalent as projective coordinate coverings. Thus an affine structure belongs to a well-defined projective structure. We say that the affine structure is *subordinate* to that projective structure. In the same way, a projective structure is subordinate to a well-defined complex structure. On a Riemann surface, which has a fixed complex structure, we will be interested in the projective and affine structures subordinate to that complex structure.

In the remainder of section 4, we will work towards defining the geometric realization of an affine or projective structure and relating it to the uniformization theorem. We will need to prove the existence of projective structures on Riemann surfaces, and for that we will need connections on a Riemann surface. Connections will also give us a handle on projective (affine) structures via a canonical correspondence. Following, we will use the group structure of projective (affine) transformations to define the coordinate cohomology class, and from there arrive at the geometric realization.

4.2 Connections

Consider any complex analytic coordinate covering $\Upsilon = \{U_\alpha, z_\alpha\}$ of a Riemann surface M. The canonical bundle $\kappa \in H^1(M, \mathcal{O})$ is defined by the cocycle $\kappa_{\alpha\beta}(p) = f'_{\alpha\beta}(z_\beta(p))^{-1}$ for coordinate transition functions $f_{\alpha\beta}$, and $p \in U_\alpha \cap U_\beta$. To each $U_\alpha \cap U_\beta$, associate the complex analytic function $\theta_\nu f_{\alpha\beta}$, defined in $z_\beta(U_\alpha \cap U_\beta)$. Denote the function by $\sigma_{\nu\alpha\beta}$; i.e., in terms of the local coordinate $z_\beta(p)$ for $p \in U_\alpha \cap U_\beta$, let

$$\sigma_{\nu\alpha\beta}(z_\beta(p)) = \theta_\nu f_{\alpha\beta}(z_\beta(p)).$$

Observe that the transition functions satisfy $f_{\alpha\gamma}(z_\gamma) = f_{\alpha\beta} \circ f_{\beta\gamma}(z_\gamma)$ in $U_\alpha \cap U_\beta \cap U_\gamma$. Using equation (4.3), we have

$$\theta_\nu f_{\alpha\gamma}(z_\gamma) = \theta_\nu f_{\alpha\beta}(z_\beta) \cdot f'_{\beta\gamma}(z_\gamma)^{-\nu} + \theta_\nu f_{\beta\gamma}(z_\gamma),$$

which can be rewritten

$$\sigma_{\nu\alpha\gamma}(z_\gamma(p)) = \sigma_{\nu\alpha\beta}(z_\beta(p)) \cdot \kappa_{\beta\gamma}(p)^{-\nu} + \sigma_{\nu\beta\gamma}(z_\gamma) \tag{4.4}$$

for $p \in U_\alpha \cap U_\beta \cap U_\gamma$. Considering $\sigma_{\nu\alpha\beta}$ as an element in $\Gamma(U_\alpha \cap U_\beta, \mathcal{O}(\kappa^\nu))$, we have

$$\sigma_{\nu\alpha\beta}(z_\gamma(p)) = \kappa^\nu_{\gamma\beta}(p)\sigma_{\nu\alpha\beta}(z_\beta(p)).$$

Then equation (4.4) becomes

$$\sigma_{\nu\alpha\gamma}(z_\gamma(p)) = \sigma_{\nu\alpha\beta}(z_\gamma(p)) + \sigma_{\nu\beta\gamma}(z_\gamma(p)).$$

This shows that $(\sigma_{\nu\alpha\beta})$ is a cocycle for the covering Υ. Therefore, to any complex analytic coordinate covering Υ of M, there is a canonically associated cocycle

$$\sigma_\nu(\theta_\nu f_{\alpha\beta}) \in Z^1(\Upsilon, \mathcal{O}(\kappa^\nu)),\ \nu = 1, 2.$$

An $\mathcal{F}_\nu$ *connection for the covering* Υ is a zero cochain $h = (h_\alpha) \in C^0(\Upsilon, \mathcal{O}(\kappa^\nu))$ such that $\delta h = \sigma_\nu$, where δ is the coboundary operator. Two connections, h, h' for the coverings Υ, Υ' are called *equivalent* if together they form part of a connection for the union of two coverings. Then an equivalence class of connections will be called an $\mathcal{F}_\nu$ *connection for the mainfold* M. Again, we use the adjectives *affine* and *projective* for the appropriate connections. Explicitly, an $\mathcal{F}_\nu$ connection for Υ consists of sections $(h_\alpha) \in \Gamma(U_\alpha, \mathcal{O}(\kappa^\nu))$ such that

$$\sigma_{\nu\alpha\beta}(p) = h_\beta(p) - h_\alpha(p) \text{ for } p \in U_\alpha \cap U_\beta. \tag{4.5}$$

In terms of the local coordinate z_α, the section h_α is a holomorphic function; the coboundary condition can be rewritten as

$$\sigma_{\nu\alpha\beta}(z_\beta(p)) = h_\beta(z_\beta(p)) - \kappa_{\alpha\beta}(p)^{-\nu} h_\alpha(z_\alpha(p)) \tag{4.6}$$

for $p \in U_\alpha \cap U_\beta$.

The following theorem relates projective connections to projective structures and will allow us to prove the existence of complex projective structures on Riemann surfaces.

Theorem 4.1. *There is a canonical one-one correspondence between the $\mathcal{F}_\nu$ connections on a Riemann surface and the $\mathcal{F}_\nu$ structures on that surface.*

Proof. Let H be an $\mathcal{F}_\nu$ connection on a Riemann surface M. Choose a representative connection $(h_\alpha) \in C^0(\Upsilon, \mathcal{O}(\kappa^\nu))$ for some complex analytic coordinate covering $\Upsilon = \{U_\alpha, z_\alpha\}$ of M. First we show that, after passing to a refinement of the covering if necessary, there are complex analytic homeomorphisms w_α on $z_\alpha(U_\alpha) = V_\alpha \subset \mathbb{C}$ such that $h_\alpha(z_\alpha) = \theta_\nu w_\alpha(z_\alpha)$. This amounts to finding a solution to the equation

$$\theta_\nu w_\alpha(z_\alpha) - h_\alpha(z_\alpha) = 0$$

in a neighborhood of a given point, with $w'_\alpha(z_\alpha) \neq 0$. In the affine case, our equation is

$$w''_\alpha - w'_\alpha h_\alpha = 0,$$

which has solutions for arbitrary values of w'_α at any point. In the projective case, we use the formula

$$\theta_2 w(z) = -2w'(z)^{1/2} \frac{d^2}{dz^2} w'(z)^{-1/2}.$$

Then we let $v = (w'(z))^{-1/2}$, or $w'(z) = v^{-2}$ so that our equation becomes

$$v''_\alpha(z) - \frac{1}{4} v_\alpha h_\alpha = 0,$$

which also has solutions with arbitrarily prescribed w_α.

Now having solved for w_α, any general such homeomorphism $\tilde{w}_\alpha$ satisfying $h_\alpha = \theta_\nu \tilde{w}_\alpha$ can be written as $\tilde{w}_\alpha = u_\alpha \circ w_\alpha$ for $u_\alpha \in \mathcal{F}_\nu$. For we can write $u_\alpha = \tilde{w}_\alpha \circ w_\alpha^{-1}$, so that $\tilde{w}_\alpha = u_\alpha \circ w_\alpha$. Then using equation (4.3), we have

$$h_\alpha = \theta_\nu \tilde{w}_\alpha = \theta_\nu u_\alpha \cdot (w'_\alpha)^\nu + \theta_\nu w_\alpha.$$

Since $w'_\alpha \neq 0$ and $\theta_\nu w_\alpha = h_\alpha$, we have $\theta_\nu u_\alpha = 0$, so that $u_\alpha \in \mathcal{F}_\nu$. Given an open cover $\{U_\alpha\}$, the most general complex analytic coordinate covering is $\{U_\alpha, w_\alpha \circ z_\alpha\}$, where z_α is a coordinate covering and $w_\alpha : V_\alpha \to W_\alpha \subset \mathbb{C}, V_\alpha = z(U_\alpha)$, are complex analytic homeomorphisms, The transition functions are given by

$$\tilde{f}_{\alpha\beta} = (w_\alpha \circ w_\alpha)(w_\beta \circ z_\beta)^{-1} = w_\alpha \circ f_{\alpha\beta} \circ w_\beta^{-1}.$$

We rewrite this as

$$\tilde{f}_{\alpha\beta} \circ w_\beta = w_\alpha \circ f_{\alpha\beta}.$$

Again we apply equation (4.3):

$$\theta_\nu \tilde{f}_{\alpha\beta} \cdot (w'_\beta)^\nu + \theta_\nu w_\beta = (\theta_\nu w_\alpha)(f'_{\alpha\beta})^\nu + \theta_\nu f_{\alpha\beta},$$

which we can rewrite as

$$(\theta_\nu \tilde{f}_{\alpha\beta})(w'_\beta)^\nu = h_\alpha (\kappa_{\beta\alpha})^\nu - h_\beta + \sigma_{\nu\alpha\beta},$$

where $h_\alpha = \theta_\nu w_\alpha$. From formula 4.6, h_α is a connection for the covering Υ if and only if $\theta_\nu \tilde{f}_{\alpha\beta} = 0$, if and only if $(\tilde{f}_{\alpha\beta})$ are transition functions of an $\mathcal{F}_\nu$ coordinate covering. The correspondence between $\mathcal{F}_\nu$ connections for Υ and $\mathcal{F}_\nu$ coordinate coverings is one-to-one, by the above observations. Equivalences are preserved, and the theorem follows. □

4.3 Existence of Projective Structures on Riemann Surfaces

The existence of a holomorphic connection for a covering on a surface is equivalent to the condition that the associated cocycle $(\sigma_{\nu\alpha\beta})$ is cohomologous to zero. The vanishing of $H^1(M, \Omega(\kappa^\nu))$ guarantees the existence of at least one connection. If a connection h exists, the most general connection is $h + g$, where $g \in \Gamma(M, \mathcal{O}(\kappa^\nu))$ is an arbitrary section, i.e., g is an arbitrary abelian ($\nu = 1$) or quadratic ($\nu = 2$) differential. *Thus there is a one to one correspondence between projective connections (if they exist) and quadratic differentials, and a one to one correspondence between affine connections (if they exist) and abelian differentials.* Now have several corollaries that show the existence of affine and projective structures on Riemann surfaces.

Corollary 4.1. *Any open RS admits a complex affine structure subordinate to the given complex structure. The set of all such structures can be put into one to one corespondence with the set of abelian differentials on the surface.*

The corollary follows from the fact that $H^1(M, \mathcal{O}(\kappa)) = 0$ for an open Riemann surface M, which can be shown by other methods.

Corollary 4.2. *A compact Riemann surface M admits affine strctures if and only if $c(\kappa) = 0$, hence if and only if the surface has genus one.*

The proof is an application of the Serre duality theorem.

Corollary 4.3. *Any Riemann surface admits a complex projective structure subordinate to the given complex structure. The set of all such structures can be put into one to one correspondences with the set of quadratic differentials on the surface.*

Proof. For an open Riemann surface, the proof is as in corollary 4.1. For a compact Riemann surface M, the case of genus 0 or 1 is obvious. For genus $g > 1$, by the Serre duality theorem, $H^1(M, \mathcal{O}(\kappa^2)) \cong \Gamma(M, \mathcal{O}(\kappa^{-1}))$. Since the Chern class $c(\kappa^{-1}) = -c(\kappa) = 2 - 2g$, we have that $c(\kappa^{-1}) < 0$, and therefore $H^1(M, \mathcal{O}(\kappa^2)) \cong \Gamma(M, \mathcal{O}(\kappa^{-1})) = 0$, and again we have the existence of projective structures. □

4.4 The Coordinate Cohomology Class

Viewing the projective transformations as complex analytic homeomorphisms $\phi : \mathbb{P} \to \mathbb{P}$, $\mathcal{P}$ is the projective linear group of rank 2 over $\mathbb{C}$. We will use the group properties to define a cohomology class that will be used in the geometric realization. In the following sections, we will be dealing with the projective case; one can find corresponding statements for the affine case.

We make use of the cohomological machinery for coefficients in an abstract group. The machinery is defined in a natural way, and we omit the details, except for two points: letting $\Upsilon = \{U_\alpha\}$ be an open covering of a topological surface M and G an abstract group, the one-cocycle condition for a one-cochain $(\phi_{\alpha\beta}) \in C^1(\Upsilon, G)$ is that $\phi_{\alpha\beta} = \phi_{\beta\alpha}^{-1}$ and $\phi_{\alpha\beta}\phi_{\beta\gamma} = \phi_{\alpha\beta}$ whenever $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$; the equivalence condition for two one-cocycles $(\phi_a b), (\psi_{\alpha\beta})$ is that there exist a zero cochain $(\theta_\alpha) \in C^0(\Upsilon, G)$ such that $\psi_{\alpha\beta} = \theta_\alpha \phi_{\alpha\beta} \theta_\beta^{-1}$.

We are interested in the first cohomology set $H^1(M, G)$, which we can relate to projective structures via the following lemma, letting $G = \mathcal{P}$.

Lemma 4.1. *There is a canonical mapping from the set of projective structures on a surface M into the cohomology set $H^1(M, \mathcal{P})$.*

Proof. For any projective structure, select a representative coordinate covering $\Upsilon = \{U_\alpha, z_\alpha\}$ with coordinate transition functions $(\phi_{\alpha\beta})$. These functions are elements of $\mathcal{P}$ and satisfy $\phi_{\alpha\beta} = \phi_{\alpha\beta}^{-1}$ and $\phi_{\alpha\gamma} = \phi_{\alpha\beta}\phi_{\beta\gamma}$ whenever $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$. Then $(\phi_{\alpha\beta})$ define a 1-cocycle in $H^1(M, \mathcal{P})$. If $\{U_\alpha, \tilde{z}_\alpha\}$ is an equivalent projective coordinate covering in terms of the same given cover $\{U_\alpha\}$, with transition functions $(\tilde{\phi}_{\alpha\beta})$, then $\tilde{z}_\alpha$ and z_α are related by functions $\theta_\alpha \in \mathcal{P}$, so that $\tilde{\phi}_{\alpha\beta} = \theta_\alpha \phi_{\alpha\beta} \theta_\beta^{-1}$. Then the cocycles $(\tilde{\phi}_{\alpha\beta}), (\phi_{\alpha\beta})$ are equivalent, and the mapping from equivalence classes of projective coordinate coverings in terms of Υ into $H^1(\Upsilon, \mathcal{P})$ is well defined. The mapping is compatible with refinement. Two projective coordinate coverings are equivalent if and only if they induce equivalent coordinate coverings for a common refinement. □

The element of $H^1(M, \mathcal{P})$ that corresponds to a projective stucture is called the *coordinate (cohomology) class* of the structure.

Remark 4.1. If restricted to projective structures subordinate to a fixed complex structure, one can show that this correspondence is one-to-one.

We saw in our first two proofs that the simple-connectedness of our surface was required in order to get the uniformizing map. Additionally, the uniformization of an arbitrary Riemann surface was achieved by considering the non-trivial fundamental group and relating it to cover transformations. We should therefore have a way to relate projective structures to the fundamental group of a surface; this relation will be via the coordinate class. Again consider an abstract group G, and let $\mathrm{Hom}(\pi_1(M,p),G)$ be the set of homomorphisms from the fundamental group into G. Let G act on $\mathrm{Hom}(\pi_1(M,p),G)$ by conjugation; i.e., an element $g \in G$ takes $\chi \in \mathrm{Hom}(\pi_1(M,p),G)$ to $\chi^g \in \mathrm{Hom}(\pi_1(M,p),G)$ defined by $\chi^g(\pi) = g^{-1}\chi(\pi)g$ for $\pi \in \pi_1(M,p)$. Then $\mathrm{Hom}(\pi_1(M,p),G)/G$ is the quotient space under the group action. Our link between projective structures and the fundamental group is provided by the following lemma.

Lemma 4.2. *For any surface M and any group G, there is a natural one to one correspondence between the cohomology set $H^1(M,G)$ and the set $\mathrm{Hom}(\pi_1(M,p),G)/G$.*

We omit the proof and proceed to the geometric realization.

4.5 Geometric Realization

Let M be a Riemann surface with projective structure. It is easy to see that this structure provides a projective structure on the covering surface $\widetilde{M}$. We now consider lemma 4.2 with the case that $G = \mathcal{P}$. Since the covering surface is simply connected, its fundamental group is 1, so the lemma implies that the first cohomology group $H^1(M,\mathcal{P})$ is trivial. Thus there is only one trivial coordinate cohomology class. Since any Riemann surface admits projective structures, there is a projective structure whose coordinate cohomology class is trivial. Then there is a representative coordinate covering of that projective structure such that the coordinate transition functions are the identity mapping. We obtain from these local coordinate maps a *global mapping* $\rho : \widetilde{M} \to D \subset \mathbb{P}$. Since ρ is a local homeomorphism, D is a connected open subset of $\mathbb{P}$. An equivalent coordinate covering representing the same projective structure defines a mapping $\rho_1 : \widetilde{M} \to D_1$, and the mappings are related by $\rho_1 = R \circ \rho$ for some projective transformation R. So ρ is unique up to projective transformations.

Now let $\pi_1(M)$ be the usual fundamental group of the Riemann surface M, and let $\Pi \cong \pi_1(M)$ be the group of cover transformations of $\widetilde{M}$. Then $\widetilde{T}$ is represented by a projective transformation in terms of the local coordinates for the given covering. This means that for any point $\tilde{p}_0 \in \widetilde{M}$, there is a projective transformation $T \in \mathcal{P}$ such that

$$\rho(\widetilde{T}\tilde{p}) = T(\rho(\tilde{p})) \tag{4.7}$$

for all points $\tilde{p}$ near $\tilde{p}_0$, since ρ is defined in terms of local mappings. Since equation (4.7) is valid in a local neighborhood of $\tilde{p}_0$, the map $\tilde{p}_0 \mapsto T$ is locally constant. Since $\widetilde{M}$ is connected, equation (4.7) is valid for all points in $\widetilde{M}$. We define the map $\rho^* : \Pi \to \mathcal{P}$ by $\rho^*(\widetilde{T}) = T$, where $\widetilde{T}$ and T are as in equation (4.7). This map is evidently a group homomorphism. Also, it is clear that T maps the domain D onto itself. The set of all such T, that is, the image of Π under ρ^*, is a group that will be denoted by Γ, and Γ is isomorphic to Π.

For two mappings derived from the same projective structure, related by $\rho_1 = R \circ \rho$,

$$\rho_1^* = R\rho^* R^{-1},$$

as one can verify. These considerations lead to the following definition.

Definition 4.1 (Geometric Realization). The pair of mappings

$$\rho : \widetilde{M} \to D \qquad\qquad \rho^* : \pi_1(\widetilde{M}) \to \mathcal{P} \tag{4.8}$$

related by

$$\rho(\widetilde{T}\tilde{p}) = \rho^*(\widetilde{T})\rho(\tilde{p}) \tag{4.9}$$

for all $\widetilde{T} \in \Pi, \tilde{p} \in \widetilde{M}$, is called a *geometric realization* of a given projective structure on M.

Note that ρ is a complex analytic local homeomorphism, and ρ^* is a group homomorphism. The equivalence of geometric realizations $(\rho, \rho^*), (\rho_1, \rho_1^*)$ is given by

$$\rho_1 = R\rho \tag{4.10}$$

$$\rho_1^* = R\rho^* R^{-1} \tag{4.11}$$

for some $R \in \mathcal{P}$. Since any geometric realization determines a projective structure, there is a one to one correspondence between projective structures on a Riemann surface and equivalence classes of geometric realizations on that surface.

Remark 4.2. If (ρ, ρ^*) is the geometric realization of a projective structure on M, the mappings ρ^* from all equivalent geometric realizations form an element $(\rho^*) \in \text{Hom}(\pi_1(M), \mathcal{P})/\mathcal{P}$. It is easy to see that this element is the image of the coordinate cohomology class of the given projective structure under the homomorphism of lemma 4.2.

What can we say about the geometric realization map ρ? We would like it to be our uniformization map, for some projective structure. We need the map to be bijective onto its image D, and we need D to be simply connected. It can be shown that the map is a covering map, which takes us part of the way there.

Theorem 4.2. *Let M be a compact topological surface of genus g ¿ 1 with a complex projective structure; let $\rho : \widetilde{M} \to D$ be its geometric realization, where $D \subset \mathbb{P}, D \neq \mathbb{P}$. Then ρ is a covering map, and either D is analytically equivalent to the unit disk, or its complement in $\mathbb{P}$ has infinitely many components.*

4.6 Problem: Proof of Classical Uniformization

If the geometrical realization of a projective structure on any Riemann surface M gives a map $\rho : \widetilde{M} \to D$ such that D is simply connected, then $\widetilde{M}$ must be simply connected and ρ must be a homeomorphism. Also, Γ must act discontinuously on D so that $M \cong D/\Gamma$. We noted above that there is a one to one correspondence between projective structures on a Riemann surface and equivalence classes of geometric realizations. Therefore, to prove the uniformization theorem using projective structures, we must answer the following question: *given any simply connected Riemann surface M, how can we choose a projective structure such that any representative geometric realization map $\rho : M \to D$ is guranteed to have the domain D be simply connected?*

Gunning did not answer this question. In light of theorem 4.1 and corollary 4.3, one could approach this problem using projective connections or quadratic differentials. Also, as we remarked after lemma 4.1, the mapping from a projective structure subordinate to the given complex structure on a Riemann surface to its coordinate cohomology class is one-to-one, so, recalling remark 4.2, the study of coordinate cohomology classes provides another possible approach to completing the proof of the uniformization theorem.

5 Conclusion

The unresolved question of obtaining uniformization by the geometric realization of a projective structure demonstrates that an old topic can still provide new problems when approached from a new perspective. From any of these three approaches, a logical further investigation would be to generalize the uniformization theorem. By our proof of the theorem, we have demonstrated that there exists some uniformization of every Riemann surface. The general uniformization problem is to find *all* uniformizations of any Riemann surface. Weyl [10] divides the problem into two parts. The first is to determine all covering surfaces of a given surface. The second is to find all possible conformal maps of all covering surfaces onto a domain of the sphere. This second problem involves the study of all conformal self-maps of the sphere, the plane, and the disk. A further generalization of the uniformization theorem to planar surfaces is also possible. Finally, a major application of uniformization is the simplification of the theory of analytic functions on a Riemann surface to functions automorphic under the group of covering transformations, and in this area there are many possibilities for subsequent study.

Acknowledgments

I would like to thank my advisor, Professor Yum-Tong Siu, for guiding my study and for helping me to gain an understanding of the subject I have written about in this thesis. I also thank my family for always supporting me.

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