THE SPRINGER CORRESONDENCE

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Contents

1. Introduction

1.1. **Background and motivation.** Let W be a finite group. It is a basic fact from representation theory (see [FH], Proposition 2.30) that the number of irreducible representations of W on finite-dimensional complex vector spaces is the same as the number of conjugacy classes in W (the proof being that both numbers count the dimension of the vector space of class functions on W). This is a wonderful statement; however, it comes with a caveat: in general there is no explicit bijection underlying this equality of numbers, and thus no way to construct an irreducible representation of W given a conjugacy class in W.

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An important and very classical exception to this caveat occurs when $\mathsf{W} = \mathsf{S_n}$, the group of permutations of a set of n elements. In this case, via cycle types, conjugacy classes in $\mathsf{S_n}$ correspond to partitions of n; these in turn correspond to Young diagrams (see [FH], page 45), which can be used to construct irreducible representations of $\mathsf{S_n}$ using Young symmetrizers ([FH] Theorem 4.3), thus giving an effective combinatorial bijection between conjugacy classes in $\mathsf{S_n}$ and irreducible representations of $\mathsf{S_n}$.

In the paper [S2], Springer gave a completely different description of this bijection, which is at its base *geometric* rather than combinatorial. It relies on the interpretation of $\mathsf{S_n}$ as the Weyl group of a certain reductive group, the group $\mathsf{GL_n}$ of invertible $n \times n$ matrices with complex coefficients. What Springer in fact constructs is an irreducible representation of $\mathsf{S_n}$ for every unipotent conjugacy class in $\mathsf{GL_n}$. These latter are also parametrized by the partitions of n through the theory of Jordan normal forms, and this procedure does give another realization of the classical bijection.

The actual construction of the representation corresponding to a unipotent conjugacy class is quite involved. To each unipotent conjugacy class one can associate a variety, the variety of Borel subgroups containing a fixed element of the class; and while the Weyl group $\mathsf{S_n}$ does not act on this variety, it remarkably *does* act on its cohomology, and the corresponding irreducible representation of $\mathsf{S_n}$ will in fact be the top-dimensional cohomology of this variety.

The goal of this paper is to give an expository account of this story and its generalization to arbitrary reductive groups, known as the *Springer correspondence*. We will use the machinery of perverse sheaves, which is particularly helpful in the crucial step—constructing the action of the Weyl group on the cohomology of the variety of Borel subgroups containing a fixed unipotent element. We have also chosen to stress functorial aspects of the Springer correspondence; more precisely, the bijection between irreducible representations of the Weyl group $\mathsf{S_n}$ and unipotent conjugacy classes in $\mathsf{GL_n}$ can be extended to an equivalence of categories between the category of finite-dimensional representations of $\mathsf{S_n}$ and the category of conjugation-equivariant perverse sheaves on the unipotent locus of $\mathsf{GL_n}$ (for a general reductive group, we get not a full equivalence, but rather an equivalence onto a Serre subcategory), and we will phrase many of our results in terms of properties of these functors.

1.2. **Notations and conventions.** As concerns type-facing, the general rule is that we will use script letters both for sheaves (say, $\mathcal{F}, \mathcal{G}$, or $\mathcal{S}$) and for varieties (say, $\mathcal{X}$, $\mathcal{Y}$, or $\mathcal{U}$). However, there are exceptions in both cases: we will use bold-facing for constant sheaves such as $\mathbf{C}$ and $\mathbf{Q}$, and when the variety is an algebraic group, we will use sans serif, say for a reductive group G or one of its Borel subgroups B. We will also use sans serif for finite groups, like the Weyl group W, and for categories, such as the category of perverse sheaves $\mathsf{P}(\mathcal{X})$ on $\mathcal{X}$. Representations (always finite-dimensional and over $\mathbf{Q}$) will be denoted by greek letters such as ρ and τ.

As for mathematical conventions, all of our geometric objects are varieties $\mathcal{X}$, not necessarily irreducible, over the field $\mathbf{C}$ of complex numbers. Our "sheaves" on $\mathcal{X}$ will be elements of $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(\mathcal{X}; \mathbf{Q}) = \mathsf{D}(\mathcal{X})$, the constructible bounded derived category of sheaves of $\mathbf{Q}$-vector spaces on $\mathcal{X}$. If $f : \mathcal{X} \to \mathcal{Y}$ is a map, we use the plain notation $f_*, f_!, f^*, f^!$ for the associated operations on this derived category. Our full subcategory of perverse sheaves $\mathsf{P}(\mathcal{X}) \subseteq \mathsf{D}(\mathcal{X})$ comes from the middle perversity ([BBD], page 63). Additionally, since our varieties are usually acted on by an algebraic group G, we will sometimes need to use G-equivariant derived categories $\mathsf{D_G}(\mathcal{X})$ and categories of G-equivariant perverse sheaves $\mathsf{P_G}(\mathcal{X})$; this formalism, which is completely analogous to its non-equivariant counterpart, will be recalled in Appendix B.

When we speak of dimension, it will always be in the sense of algebraic geometry, not complex topology. The symbol $\bullet$ will denote the variety which is just a point with sheaf of rings $\mathbf{C}$. If $i : \mathcal{Y} \to \mathcal{X}$ is a closed immersion, then i_* is an equivalence onto a localizing triangulated subcategory, and is exact both in the perverse sense and the normal sense; so we will never feel bad about dropping it from the notation, and viewing a sheaf on $\mathcal{Y}$ also as a sheaf on $\mathcal{X}$ without comment.

If P is a category and $\mathcal{F}$ and $\mathcal{G}$ are objects of P, we will write $\mathsf{P}(\mathcal{F}, \mathcal{G})$ for the set of morphisms from $\mathcal{F}$ to $\mathcal{G}$ in P. If, furthermore, P is abelian, we will denote by $\mathsf{P}^n(\mathcal{F}, \mathcal{G})$ what is usually called $\mathrm{Ext}^n_{\mathsf{P}}(\mathcal{F}, \mathcal{G})$, the Yoneda Ext group; on the other hand, if D is triangulated, we set $\mathsf{D}^n(\mathcal{F}, \mathcal{G}) = \mathsf{D}(\mathcal{F}, \mathcal{G}[n])$. Recall that if D is any of the usual derived categories of P (i.e. bounded or unbounded however one likes), these notations are consistent.

1.3. **Statement of the main results.** Let G be a connected reductive group over $\mathbf{C}$, and let $\mathcal{U}$ be the unipotent locus of G, which is a closed subvariety carrying the conjugation action of G. In Section 2.1, we will give the definition and basic geometric properties of a certain G-equivariant resolution of singularities $p : \widetilde{\mathcal{U}} \to \mathcal{U}$, called the *Springer map.* The first result of note is the following, to be proved in Section 3.2:

Proposition 1.1. *The sheaf $\mathcal{S} = p_*\mathbf{Q}[\dim(\mathcal{U})]$ (called the* Springer sheaf*) is perverse, and carries a natural right action of the Weyl group W of G.*

In Appendix A, we show how such an object with such an action gives rise to a $\mathbf{Q}$-linear functor ${}_{\mathsf{W}}\mathsf{Rep} \to \mathsf{P}_{\mathsf{G}}(\mathcal{U})$ from the category of finite-dimensional representations of W to the category of G-equivariant perverse sheaves on $\mathcal{U}$; this will be called the *Springer functor.* Here is the main theorem:

Theorem 1.2. *The Springer functor identifies ${}_{\mathsf{W}}\mathsf{Rep}$ with a Serre subcategory of $\mathsf{P}_{\mathsf{G}}(\mathcal{U})$.*

Now, consider a parabolic subgroup P of G, with Levi factor L. In section 3.2, we will construct functors $\mathrm{Res}^{\mathsf{G}}_{\mathsf{L}} : \mathsf{D}_{\mathsf{G}}(\mathcal{U}_{\mathsf{G}}) \to \mathsf{D}_{\mathsf{L}}(\mathcal{U}_{\mathsf{L}})$ and $\mathrm{Ind}^{\mathsf{G}}_{\mathsf{L}} : \mathsf{D}_{\mathsf{L}}(\mathcal{U}_{\mathsf{L}}) \to \mathsf{D}_{\mathsf{G}}(\mathcal{U}_{\mathsf{G}})$, which fit into the following theorem concerning functorial properties of the Springer functor:

Theorem 1.3. *The Springer functor likes to intertwine operations on representations and operations on sheaves:*

(1) It intertwines duality on representations and Verdier duality D on sheaves;
(2) Let L be a Levi factor as above, with associated inclusion $\mathsf{W}_{\mathsf{L}} \subseteq \mathsf{W}_{\mathsf{G}}$ of Weyl groups. Then:
  (a) It intertwines $\mathrm{Res}^{\mathsf{W}_{\mathsf{G}}}_{\mathsf{W}_{\mathsf{L}}}$ and the above $\mathrm{Res}^{\mathsf{G}}_{\mathsf{L}}$;
  (b) It intertwines $\mathrm{Ind}^{\mathsf{W}_{\mathsf{G}}}_{\mathsf{W}_{\mathsf{L}}}$ and the above $\mathrm{Ind}^{\mathsf{G}}_{\mathsf{L}}$.
(3) Let G' be another connected reductive group, with uninpotent locus $\mathcal{U}'$ and Weyl group W'. Then $\mathsf{G} \times \mathsf{G}'$ has unipotent locus $\mathcal{U} \times \mathcal{U}'$ and Weyl group $\mathsf{W} \times \mathsf{W}'$, and the Springer functor intertwines the external product $\boxtimes$ on representations and the external product $\boxtimes$ on perverse sheaves.

We will also consider more closely the case $\mathsf{G} = \mathsf{GL_n}$ (with Weyl group $\mathsf{W} = \mathsf{S_n}$), which affords many simplifications.

Theorem 1.4. *For $\mathsf{G} = \mathsf{GL_n}$, the Springer functor is an equivalence of categories.*

In Appendix B.4, we prove a result which implies that, with $\mathsf{G} = \mathsf{GL_n}$, the category $\mathsf{P}_{\mathsf{G}}(\mathcal{U})$ is artinian and noetherian, and has only finitely many (isomorphism classes of) simple objects, these being moreover in bijection with the unipotent conjugacy classes in $\mathsf{GL_n}$. We will then have:

Theorem 1.5. *The Springer functor induces, on isomorphism classes of simple objects, a bijection between the irreducible representations of* $\mathsf{S_n}$ *and the unipotent conjugacy classes in* $\mathsf{GL_n}$.

1.4. **Acknowledgements.** I would like to thank my advisor, Dennis Gaitsgory, for introducing and teaching this beautiful subject to me. Everything I've written here comes from something he has told me, and I'm very grateful for the guidance I've received and the things I've learned. He also suggested the wonderful books [S] and [BBD], whose authors I would also like to thank: both books have been invaluable both as learning aids and as references.

On a more personal note, my friends and family deserve my deepest thanks for their love, their patience, and their support.

2. The geometry of the unipotent locus

We start by introducing our geometric playground: the unipotent locus of a reductive group. We will see that this variety has a very natural resolution of singularities, called the Springer map, which, besides enjoying a number of interesting and simple properties, is fundamental for the Springer correspondence.

2.1. **Fundamental facts.** Let G be a connected reductive group. We denote by $\mathcal{U}_\mathsf{G}$, or just by $\mathcal{U}$ when the group is clear, the unipotent locus of G. It is a closed subvariety, and we will always consider it together with its conjugation action by G. Let also B be a fixed Borel subgroup of G, and U its unipotent radical, which is the same as its unipotent locus. It is a normal subgroup of B, and we consider it together with its B-action by conjugation. We let T be a fixed maximal torus in B, though we will often want to think of T as B/U (see [S], Theorem 6.3.5). We let $\mathsf{W} = \mathsf{N_G}(\mathsf{T})/\mathsf{T}$ denote the Weyl group of G, and $\dot{\mathsf{W}} \subseteq \mathsf{N_G}(\mathsf{T})$ some fixed set of representatives for W (nothing we ever do will depend on these representatives). Given $w \in \mathsf{W}$ with representative $\dot{w}$ and a closed subgroup H of G, let $\mathsf{H}^w = \mathsf{H} \cap \dot{w}^{-1}\mathsf{B}\dot{w}$. In particular we have B^w and U^w (not to be confused with the $\mathsf{U}_w = \mathsf{U} \cap \dot{w}^{-1}\mathsf{U}^-\dot{w}$ of [S]); these are closed, connected, solvable subgroups of G, and we have $\mathsf{B}^w \simeq \mathsf{T} \times \mathsf{U}^w$ (via multiplication; see [S] Corollary 8.3.10 and Theorem 6.3.5).

Recall that every element of $\mathcal{U}$ can be conjugated into U, so if we are only interested in G-conjugacy classes in $\mathcal{U}$, then U is in some sense a reasonable smooth approximation. It is pretty nice, since it's even an affine space; however, it has a defect: it carries only a B-action, and not a G-action. So, to obtain a better smooth approximation to $\mathcal{U}$, we consider instead the base-change of the B-space U up to the G-space $\mathsf{G} \times^\mathsf{B} \mathsf{U}$ (c.f. Corollary B.10); it turns out that this will provide a G-equivariant resolution of singularities for $\mathcal{U}$, and many other things besides:

Definition 2.1. *Notations as above,* G *being a reductive group. Define* $\widetilde{\mathcal{U}} = \mathsf{G} \times^\mathsf{B} \mathsf{U}$. *The* ***Springer map*** *is the morphism* $p : \widetilde{\mathcal{U}} \to \mathcal{U}$ *given by* $[g, u] \mapsto gug^{-1}$*; we think of it as the* G*-replacement for the inclusion* $\mathsf{U} \to \mathcal{U}$.

The following proposition collects the fundamental algebraic-geometric properties of p.

Proposition 2.2. *We have the following:*

(1) The unipotent locus $\mathcal{U}$ is irreducible of dimension $2\dim(\mathsf{U})$, and has only finitely many G-orbits;

(2) For $u \in \mathcal{U}$, the fiber $\mathcal{B}_u := p^{-1}(u)$ can be identified with the set of Borel subgroups of G containing u. It is equidimensional and connected; furthermore, if $\mathcal{O}_u$ denotes the orbit (i.e. conjugacy class) of u, we have

$$\dim(\mathcal{O}_u) + 2\dim(\mathcal{B}_u) = \dim(\mathcal{U}).$$

(3) The Springer map p is a resolution of singularities, i.e. it is proper and birational, and $\widetilde{\mathcal{U}}$ is smooth; it is furthermore G-equivariant (by definition).

(4) If $\mathcal{X}$ denotes the variety-theoretic $\mathsf{U} \times_{\mathcal{U}} \widetilde{\mathcal{U}}$, then $\mathcal{X}$ admits a stratification $\mathcal{X} = \coprod_{w \in \mathsf{W}} \mathcal{X}(w)$ parametrized by the Weyl group, and we have $\mathcal{X}(w) \simeq \mathsf{B} \times^{\mathsf{B}^w} \mathsf{U}^w$. Thus, the $\mathcal{X}(w)$ are all smooth and connected of dimension $\dim(\mathsf{U})$; so their closures are the components of $\mathcal{X}$, and $\mathcal{X}$ is equidimensional of dimension $\dim(\mathsf{U})$.

We make a few remarks before starting the proof. Firstly, we warn the reader right away that the order of the proof will not come close to following the order of the statements; it will jump around like crazy. We will start with a different description of $\widetilde{\mathcal{U}}$, related to point (2) and point (3), then proceed to point (4), which we will essentially use to prove the rest.

Secondly, we will not prove the connectedness and equality of dimensions in (2); instead we will prove only the equidimensionality and an inequality $\leq$ of dimensions, and refer to the paper of Spaltenstein [Sp] for the missing statements.[1]

Thirdly, we remark that, in this proof, we will be claiming that a lot of squares are pullbacks in the category of schemes, mostly in order to apply faithfully flat descent. It will always obviously be so in the category of varieties; the only question will be whether there can be nilpotents. However, each of the squares will deal with pullback by a smooth map (coming from a map of homogeneous G-spaces; see [S] Theorem 4.3.6), so that the scheme-theoretic pullback is necessarily reduced, thanks to the following lemma:

Lemma 2.3. *Let $f : \mathcal{Y} \to \mathcal{Z}$ be a smooth morphism of schemes, with $\mathcal{Z}$ locally noetherian. If $\mathcal{Z}$ is reduced, so is $\mathcal{Y}$.*

Proof. We easily reduce to the affine case, say $\mathcal{Y}= \operatorname{Spec}(B)$ and $\mathcal{Z}= \operatorname{Spec}(A)$, with A noetherian. Denote by k_i the residue fields of the (finitely many) minimal primes of A; then we have an injection $A \hookrightarrow \oplus_i k_i$. Tensoring with B (flat over A by smoothness) gives an injection $B \hookrightarrow \oplus_i B \otimes k_i$. But $B \otimes k_i$ is smooth over the field k_i, hence reduced; the injection then shows that B is as well. □

We now turn to the proof of the proposition.

Proof. As promised, we start with a different description of $\widetilde{\mathcal{U}}$, more precisely as a certain closed subvariety of $\mathcal{B} \times \mathcal{U}$, where $\mathcal{B} = \mathsf{G}/\mathsf{B}$ is the flag variety. Consider the map $\widetilde{\mathcal{U}} \to \mathcal{B} \times \mathcal{U}$

[1] A remark concerning the importance of these statements whose proofs were omitted. One should note that, in the proof below of Proposition 2.2, we use them in order to deduce that $\mathcal{U}$ has only finitely many orbits (part of (1)) and that p is birational (part of (3)), so in some sense we are not really including the proofs of those facts either. The birationality of p will play no role for us; it is included for its independent interest. However, $\mathcal{U}$ having only finitely many orbits is fundamental, being invoked in the proof of Proposition 1.1 (the Springer sheaf is perverse). The equality of dimensions in (2) will also be important in and of itself, since it will give us Springer's version of his correspondence (Theorem 4.2), which in turn will let us deduce *without counting* that the Springer functor is an equivalence of categories in the case $\mathsf{G} = \mathsf{GL_n}$—this is, of course, the culmination of the paper.

given by $[g,u] \mapsto (g\mathsf{B}, gug^{-1})$. It is a closed immersion, since after faithfully flat base-change by the quotient $\mathsf{G} \to \mathcal{B}$ it becomes $id_{\mathsf{G}} \times i$, with $i : \mathsf{U} \to \mathcal{U}$ the inclusion. Furthermore, its image is easily identified as those $(g\mathsf{B}, u)$ for which $u \in g\mathsf{B}g^{-1}$, and in these terms p is just the projection to the second factor. This makes the first sentence of (2) clear, since, recall, $g\mathsf{B} \leftrightarrow g\mathsf{B}g^{-1}$ gives a bijection between $\mathcal{B}$ and the set of Borel subgoups of G (see [S] 6.4.13). We also see that p is proper (part of claim (3)), being a closed immersion followed by projection from the projective $\mathcal{B}$.

Now we consider (4). By the above interpretation of $\widetilde{\mathcal{U}}$, we can view $\mathcal{X}$ as the set of $(u, g\mathsf{B})$ with $u \in \mathsf{U} \cap g\mathsf{U}g^{-1}$; thus $\mathcal{X}$ admits a natural projection to $\mathcal{B}$. However, Bruhat's lemma ([S], Theorem 8.3.8) gives us a stratification of $\mathcal{B}$ into its B-orbits, parametrized by the $w \in \mathsf{W}$:

$$\mathcal{B} = \coprod_{w \in \mathsf{W}} \mathsf{B} \cdot (\dot{w}^{-1}\mathsf{B}),$$

which pulls back to a stratification $\mathcal{X} = \coprod_{w \in W} \mathcal{X}(w)$ of $\mathcal{X}$, each piece of which fits into a pullback square

$$\begin{tikzcd} \mathsf{B} \times \mathsf{U}^w \arrow[r] \arrow[d] & \mathcal{X}(w) \arrow[d] \\ \mathsf{B} \arrow[r] & \mathsf{B} \cdot (\dot{w}\mathsf{B}), \end{tikzcd}$$

where the left map is projection and the bottom map is the faithfully flat action map, which has stabilizer B^w; we deduce from descent that that the right-hand map is smooth, hence so is $\mathcal{X}(w)$; then the pullback square again gives us the desired property.

This finishes (4); we now turn to (1). Firstly, since $\widetilde{\mathcal{U}} \to \mathcal{B}$ definitionally admits $\mathsf{G} \times \mathsf{U} \to \mathsf{G}$ as a faithful flat base-change, we see at once that $\widetilde{\mathcal{U}}$ is smooth, irreducible, and of dimension $2\dim(\mathsf{U})$. Then $\mathcal{U}$ is irreducible as well, since p is surjective (every unipotent element can be conjugated into U). Now, since p is proper, it has some generic relative dimension; the situation being G-equivariant, this must be the same as the generic relative dimension of $\mathcal{X} \to \mathsf{U}$, which we've seen is zero. Thus we have $\dim(\mathcal{U}) = \dim(\widetilde{\mathcal{U}}) = 2\dim(\mathsf{U})$, and this finishes (1), except for the claim about finitely many orbits.

We leave this aside for now, and move on to (2). For $u \in \mathcal{U}$, let $\widetilde{\mathcal{O}_u} = p^{-1}(\mathcal{U})$. We have $\widetilde{\mathcal{O}_u} = \mathsf{G} \times^{\mathsf{B}} (\mathcal{O}_u \cap \mathsf{U})$, by construction (see the remarks preceding Proposition B.10). On the other hand, we have the pullback

$$\begin{tikzcd} \mathsf{G} \times \mathcal{B}_u \arrow[r] \arrow[d] & \widetilde{\mathcal{O}_u} \arrow[d] \\ \mathsf{G} \arrow[r] & \mathcal{O}_u, \end{tikzcd}$$

exhibiting $\mathsf{G} \times \mathcal{B}_u$ as an equidimensional flat base-change of $\widetilde{\mathcal{O}_u}$; putting things together, we deduce that $\mathcal{B}_u$ is equidimensional, and that

$$\dim(\mathcal{O}_u) + \dim(\mathcal{B}_u) = \dim(\mathcal{O}_u \cap \mathsf{U}) + \dim(\mathsf{U}).$$

On the other hand, from the dimension claim in (4), we have

$$\dim(\mathcal{O}_u \cap \mathsf{U}) + \dim(\mathcal{B}_{\mathsf{u}}) \leq \dim(\mathsf{U});$$

summing the above two lines and using the dimension claim in (1) gives the inequality $\leq$ in (2), which is all we will show; see the remarks preceding this proof.

However, we now use *equality* in (2) ([Sp]) to see there are only finitely many unipotent conjugacy classes, finishing (1): if u runs over a set of representatives of the conjugacy classes, the inverse images of the $\mathcal{O}_u$ in $\mathcal{X}$ are equidimensional of maximal possible dimension in $\mathcal{X}$ (because we have equality in the above line, not just $\leq$), and thus each gives a subset of the components; clearly these subsets are disjoint, but there are only finitely many components to go around, and so only finitely many $\mathcal{O}_u$.

The reader armed with a checklist will know that only the birationality of p remains. For this, since $\mathcal{U}$ is irreducible and has only finitely many orbits, there is an open orbit $\mathcal{O}$; take a $u \in \mathcal{O}$. Then we have a pullback

$$\begin{tikzcd} \mathsf{G} \times \mathcal{B}_\mathsf{u} \arrow[r] \arrow[d] & \widetilde{\mathcal{O}} \arrow[d] \\ \mathsf{G} \arrow[r] & \mathcal{O}, \end{tikzcd}$$

where $\widetilde{\mathcal{O}} = p^{-1}(\mathcal{O})$; counting dimensions gives $\dim(\mathcal{B}_u) = 0$, but the connectedness claim in (2) gives that $\mathcal{B}_u$ is connected; thus it is trivial[2], and the left map is an isomorphism; then by descent, so is the right map, as desired. □

2.2. **Interlude: introduction of the Springer sheaf.** We will soon return to geometry, but we now pause to introduce our lead actor, our principal algebraic object: the Springer sheaf $\mathcal{S}$.

Recall that G is a connected reductive group, and $\mathcal{U}$ is its unipotent locus, which carries a G-action by conjugation. Recall also the G-equivariant Springer map (resolution) $p : \widetilde{\mathcal{U}} \to \mathcal{U}$ from the previous section.

Definition 2.4. *The* ***Springer sheaf*** $\mathcal{S}_\mathsf{G}$ *(or just $\mathcal{S}$) is the element* $p_*\mathbf{Q}[\dim(\mathcal{U})]$ *of* $\mathsf{D}(\mathcal{U})$.

The following is fundamental:

Proposition 2.5. *We have* $\mathcal{S} \in \mathsf{P}_\mathsf{G}(\mathcal{U})$.

Proof. It is clearly G-equivariant; the claim is that it is perverse. Since $\widetilde{\mathcal{U}}$ is smooth of dimension $\dim(\mathcal{U})$, the sheaf $\mathbf{Q}[\dim(\mathcal{U})]$ on $\widetilde{\mathcal{U}}$ is self-dual; since p is proper, $p_* = p_!$, and so $\mathcal{S}$ is self-dual as well. Thus, it will suffice to check just one half of the perversity condition (see [BBD] page 63); so what we need is that if $j : \mathcal{Z} \to \mathcal{U}$ is a stratum, then the stalk at any $z \in \mathcal{Z}$ of the cohomology of $\mathcal{S}$ vanishes in degrees above $-\dim(\mathcal{Z})$. However, given any stratification, by refining if necessary we can assume it finer than the stratification of $\mathcal{U}$ by orbits, so say $\mathcal{Z} \subseteq \mathcal{O}_u$. But then certainly $\dim(\mathcal{Z}) \leq \dim(\mathcal{O}_u)$, and so the dimension estimate in Proposition 2.2 (2) shows that for all $x \in \mathcal{Z}$, we have

$$\dim(\mathcal{Z}) + 2\dim(\mathcal{B}_x) \leq \dim(\mathcal{U}).$$

On the other hand, by proper base change, the stalk of $\mathcal{S}$ at x is just the cohomology complex of $\mathcal{B}_x$ shifted by $\dim(\mathcal{U})$; so the above dimension estimate is exactly what's required to conclude. □

[2]It is somewhat of a cheat to invoke connectedness of the $\mathcal{B}_u$ to prove that the generic $\mathcal{B}_u$ is trivial, since the latter fact is easier than the former. However, as mentioned in the previous footnote, birationality of p will play absolutely no role for us in the remainder of this paper, so we do not feel bad about cheating during its proof, at least provided we include a footnote explaining our cheating.

Our next goal is to construct a canonical right action of W on $\mathcal{S}$, in order to be able to use the formalism of Appendix A to obtain a functor ${}_{\mathsf{W}}\mathsf{Rep} \to \mathsf{P}_{\mathsf{G}}(\mathcal{U})$. For this we will need to revisit and extend the geometry of the previous section.

2.3. **An extension of the Springer map.** The idea behind the construction of the action of W on $\mathcal{S}$ is to view the Springer map $p : \widetilde{\mathcal{U}} \to \mathcal{U}$ as a special locus of a more general map $\widetilde{\mathsf{G}} \to \mathsf{G}$, which is itself generically Galois with group W. The obvious action of W on the pushforward of the constant sheaf $\mathbf{Q}$ on this Galois locus will extend to the whole of G, and then restrict to $\mathcal{U}$.

We define $\widetilde{\mathsf{G}} := \mathsf{G} \times^{\mathsf{B}} \mathsf{B}$ (recall, the B-action on B is conjugation); it admits a map $\overline{p} : \widetilde{\mathsf{G}} \to \mathsf{G}$ given by $[g, b] \mapsto gbg^{-1}$. Here is the analog to Proposition 2.2:

Proposition 2.6. *The map $\overline{p} : \widetilde{\mathsf{G}} \to \mathsf{G}$ has the following properties:*

(1) $\widetilde{\mathsf{G}}$ is smooth, $\overline{p}$ is proper, and for $x \in \mathsf{G}$, the fiber $\mathcal{B}_x := q^{-1}(x)$ can be identified with the set of Borel subgroups of G containing x;

(2) There is a diagram

$$\begin{tikzcd} \widetilde{\mathcal{U}} \arrow[r] \arrow[d, "p"] & \widetilde{\mathsf{G}} \arrow[d, "\overline{p}"] & \widetilde{\mathcal{R}} \arrow[l] \arrow[d, "p^\circ"] \\ \mathcal{U} \arrow[r, "i"] & \mathsf{G} & \mathcal{R} \arrow[l, "j"'] \end{tikzcd}$$

where both squares are cartesian (in varieties), i is the obvious closed immersion, j is the inclusion of a G-stable dense open subset, p° is connected Galois with group W, and all maps are G-equivariant;

(3) If $\mathcal{Y}$ denotes the variety-theoretic $\mathsf{B} \times_{\mathsf{G}} \widetilde{\mathsf{G}}$, then $\mathcal{Y}$ admits a stratification $\mathcal{Y} = \coprod_{w \in \mathsf{W}} \mathcal{Y}(w)$ parametrized by the Weyl group, and we have $\mathcal{Y}(w) \simeq \mathsf{B} \times^{\mathsf{B}^w} \mathsf{B}^w$; furthermore, over $\mathcal{R}$, the map $\mathcal{Y} \to \mathsf{B}$ is constant Galois with group W.

Proof. The proofs of (1) and all but the last part of (3) are exactly as in Proposition 2.2; it suffices to replace $\mathcal{U}$ by G and U by B in the first two paragraphs of that proof. The claims about the left-hand square in (2) are also clear from the identification of $\widetilde{\mathcal{U}}$, respectively $\widetilde{\mathsf{G}}$, as a closed subvariety of $\mathcal{B} \times \mathcal{U}$, respectively $\mathcal{B} \times \mathsf{G}$. What remain are the claims concerning the open locus $\mathcal{R}$, to whose definition we now turn.

Let $\mathcal{R}$ be the set of $x \in \mathsf{G}$ whose connected centralizer $\mathsf{Z}(x)^\circ$ is a maximal torus. We will see that each $x \in \mathcal{R}$ is semi-simple; $\mathcal{R}$ is called the *regular semi-simple locus*, and is clearly G-stable. We first check that $\mathcal{R}$ is nonempty and open. For this, let $\mathcal{R}_{\mathsf{T}} \subseteq \mathsf{T}$ denote the complement to all the kernels of the roots of G (see [S] 7.1.1 and Corollary 8.1.12). These are finitely many proper closed subvarieties, so $\mathcal{R}_{\mathsf{T}}$ is nonempty and open in T. I claim that $\mathcal{R} = \mathsf{G} \setminus \overline{p}(\widetilde{\mathsf{G}} \setminus q(f^{-1}(\mathcal{R}_{\mathsf{T}})))$, where $f : \mathsf{G} \times \mathsf{B} \to \mathsf{B} \to \mathsf{T}$ is the projection and $q : \mathsf{G} \times \mathsf{B} \to \widetilde{\mathsf{G}}$ is the quotient map. Since $\overline{p}$ is proper and q is flat, this would imply that $\mathcal{R}$ is open—and nonemptiness will become clear in the course of the proof of the claim, which we now begin.

In more down-to-earth terms, the claim is exactly that $\mathcal{R}$ is the set of $x \in \mathsf{G}$ all of whose conjugates lying in B are of the form tu, with $t \in \mathcal{R}_{\mathsf{T}}$ and $u \in \mathsf{U}$. Since both $\mathcal{R}$ and this set it is claimed to be equal to are conjugation stable, it suffices to show that a $tu \in \mathsf{TU} = \mathsf{B}$ is in $\mathcal{R}$ if and only if $t \in \mathcal{R}_{\mathsf{T}}$. We note first that $\mathcal{R}_{\mathsf{T}} \subseteq \mathcal{R}$, by [S], Exercise 8.1.12 (3). We also note that, on general grounds, the semi-simple part of such a tu is conjugate to t. Now, if $t \in \mathcal{R}_{\mathsf{T}}$, then $t \in \mathcal{R}$, and so the semi-simple part of tu will be in $\mathcal{R}$; but its unipotent part commutes with its semi-simple part, so by the definition of $\mathcal{R}$ it will be trivial (see [S] Threorem 6.3.5

(ii), which says that centralizers of semi-simple elements in solvable groups are connected), and so tu equals its semi-simple part, and lies in $\mathcal{R}$. Similarly for "only if": the unipotent part of tu commutes with tu, and thus is trivial, so tu is equal to its semi-simple part, which is conjugate to t and thus lies in $\mathcal{R}$.

So $\mathcal{R}$ is open and nonempty. We now make a few important observations about $\mathcal{R}$. Firstly, every $x \in \mathcal{R}$ is contained in a *unique* maximal torus—indeed, each maximal torus containing x lies in the centralizer $\mathsf{Z}(x)$. This, together with [S], Corollary 6.4.12, shows that, for $x \in \mathcal{R}_\mathsf{T}$, conjugation by the $\dot{w}$ gives a simply-transitive action of W on the set $\mathcal{B}_x$ of Borel subgroups of G containing x. This action in fact extends to an action of W on the whole map $p^\circ : \widetilde{\mathcal{R}} \to \mathcal{R}$; in terms of the interpretation of $\widetilde{\mathsf{G}}$ as the set of pairs $(x, g\mathsf{B})$, it is simply $(x, g\mathsf{B}) \mapsto (x, g\dot{w}^{-1}\mathsf{B})$—but in order to describe it in a way which is obviously well-defined, we use the identification

$$\widetilde{\mathcal{R}} \simeq \mathsf{G} \times^\mathsf{B} (\mathcal{R} \cap \mathsf{B}) \simeq \mathsf{G} \times^\mathsf{T} \mathcal{R}_\mathsf{T} \simeq \mathsf{G}/\mathsf{T} \times \mathcal{R}_\mathsf{T},$$

the first isomorphism coming from $\mathcal{R}$ being G-stable, the second coming from the fact that each element of $\mathcal{R} \cap \mathsf{B}$ can be uniquely conjugated into $\mathcal{R}_\mathsf{T}$ by an element of U, and the last coming from the conjugation action of T on $\mathcal{R}_\mathsf{T}$ being trivial. Then in terns of $\mathsf{G}/\mathsf{T} \times \mathcal{R}_\mathsf{T}$, the action is $(gT, x) \mapsto (g\dot{w}^{-1}T, \dot{w}t\dot{w}^{-1})$. Chasing it through on elements of $\mathcal{R}_T$, we see that this recovers the above simply-transitive action.

So all that remains to show that p° is connected Galois with group W is to see that it is étale. For this, by dimension considerations, it suffices to see that the induced map on tangent spaces at any point is surjective; this is something we can check after composing with the smooth $\mathsf{G} \times \mathsf{B} \to \widetilde{\mathsf{G}}$, so we just need that, for every $x_0 \in \mathsf{B} \cap \mathcal{R}$ and $g_0 \in G$, the conjugation map $\mathsf{G} \times \mathsf{B} \to \mathsf{G}$ is smooth at (g_0, x_0). Because of G-equivariance, we can assume $g_0 = e$, the identity; so we may as well show that the map $\mathsf{G} \times \mathsf{B} \to \mathsf{G}$ given by $(x, b) \mapsto xbx^{-1}x_0^{-1}$ is smooth at (e, x_0). However, restricting to $\{e\} \times \mathsf{B}$ gives that the image of the map on tangent spaces contains the Lie algebra $L(\mathsf{B})$ of B, and restricting to $\mathsf{G} \times \{x_0\}$ gives that this image also contains a complementary subspace to $L(\mathsf{Z}(x)) = L(\mathsf{T})$, by [S] Corollary 5.4.5; this is enough to guarantee that the map on tangent spaces is surjective, as desired.

What remains is the last claim in (3). But this is simple: when $\mathcal{Y}(w)$ is identified with $\mathsf{B} \times^{\mathsf{B}^w} \mathsf{B}^w$, the regular semi-simple locus in $\mathcal{Y}(w)$ is identified with

$$\mathsf{B} \times^{\mathsf{B}^w} (\mathcal{R} \cap \mathsf{B}^w) \simeq \mathsf{B}/\mathsf{T} \times \mathcal{R}_\mathsf{T},$$

just as above; and it's easy to see that, under these identifications, the map to $\mathcal{R} \cap \mathsf{B}$ is just conjugation $(bT, t) \mapsto btb^{-1}$. We also remark that, under these identifications, the Weyl group action is just $(bT, t) \mapsto (bT, t)$—note, though: mapping the $\mathcal{Y}(w')$ component to the $\mathcal{Y}(ww')$ component. ◻

We will need the following corollary before we return to the Springer sheaf.

Corollary 2.7. *There exists a stratification of* G *with the following properties:*

(1) $\mathcal{R}$ *is a stratum;*
(2) If $\mathcal{Z}$ *is a stratum which is not* $\mathcal{R}$*, then for all* $x \in \mathcal{Z}$*, we have* $\dim(\mathcal{Z}) + 2\dim(\mathcal{B}_x) < \dim(\mathsf{G})$.

Proof. Because the map $\overline{p}$ is proper, we have a stratification $\mathsf{G} = \coprod_{n \geq 0} \mathcal{Z}_n$ into locally closed pieces $\mathcal{Z}_n = \{x \in \mathsf{G} \mid \dim(\mathcal{B}_x) = n\}$. Replace $\mathcal{Z}_0$ with $\mathcal{Z}_0 \setminus \mathcal{R}$, and add $\mathcal{R}$; this gives the desired stratification. To verify (2), we can proceed exactly as in the proof of the dimension estimate of Proposition 2.2 (2): since the $\mathcal{Z}_n$ are G-stable, we get

$$\dim(\mathcal{Z}_n) + n = \dim(\mathcal{Z}_n \cap \mathsf{B}) + \dim(\mathsf{U});$$

but pulling back to $\mathcal{Y}$ gives $\dim(\mathcal{Z}_n \cap \mathsf{B}) + n \leq \dim(\mathsf{B})$, so we deduce

$$\dim(\mathcal{Z}_n) + 2n \leq \dim(\mathsf{G}).$$

But, actually, since we're missing $\mathcal{R}$, which is open and dense in $\mathcal{Y}$ by (3) in the above proposition, we can replace the non-strict inequality with a strict one, and this gives the required estimate. □

We are now in a position to construct the action of W on the Springer sheaf $\mathcal{S}$. Here are the sheaf-theoretic consequences of the above geometric considerations.

Corollary 2.8. *The derived pushforward p°_* agrees with the usual pushforward of sheaves. The sheaf $p^\circ_*\mathbf{Q}$ carries a canonical right action of W, and its shift $\mathcal{S}^\circ := p^\circ_*\mathbf{Q}[\dim(\mathsf{G})]$ lies in $\mathsf{P}_\mathsf{G}(\mathcal{R})$.*

Proof. The first claim just comes from the usual pushforward being exact, which holds because p° is étale. The second claim is then clear: $\mathbf{Q}$ is just the sheaf of locally constant functions from $\widetilde{\mathcal{R}}$ to $\mathbf{Q}$, and so the left action of W on p° gives, by function composition, a right action of W on $p^\circ_*\mathbf{Q}$. For the third claim, G-equivariance is clear, and perversity follows from étale pushforwards also being t-exact ([BBD] Corollary 2.2.6 (i); c.f. the proof of Proposition 2.5). □

Corollary 2.9. *The Springer sheaf $\mathcal{S}$ can be obtained from $p^\circ_*\mathbf{Q}$ by a multi-step functorial operation: if we let $\overline{\mathcal{S}} = \overline{p}_*\mathbf{Q}[\dim(\mathsf{G})] \in \mathsf{D}(\mathsf{G})$, then we have $\overline{\mathcal{S}} = j_{!*}(\mathcal{S}^\circ)$, and $\mathcal{S} = i^*\overline{\mathcal{S}}[-\dim(\mathsf{T})]$. Hence $\mathcal{S}$ also carries a right action of W.*

Proof. We first note that $\overline{\mathcal{S}}$ is perverse. Indeed, this follows from the exact same argument as Proposition 2.5, using the stratification of Corollary 2.7 rather than the stratification by orbits (which would not be a stratification in this case). And in fact, the strict inequality in (2) of that Corollary, together with the characterization ([BBD] Proposition 2.1.17) of the Goresky-Macpherson extension of a self-dual sheaf, shows that $\overline{\mathcal{S}}$ is the Goresky-Macpherson extension of its restriction to $\mathcal{R}$. To conclude, it suffices to remark that $j^*\overline{\mathcal{S}} = \mathcal{S}^\circ$ and $i^*\overline{\mathcal{S}} = \mathcal{S}[\dim(\mathsf{T})]$, both by proper base-change. □

3. The Springer functor

3.1. **Beginning of proof of Theorem 1.2.** Recall that G is a connected reductive group over $\mathbf{C}$, and $\mathcal{U}$ is its unipotent locus, acted on by G through conjugation. By Corollary 2.9, the Weyl group acts on the right on the Springer sheaf $\mathcal{S}$ (Definition 2.4); by Proposition 2.5, $\mathcal{S}$ lives in the $\mathbf{Q}$-linear abelian category $\mathsf{P}_\mathsf{G}(\mathcal{U})$, which clearly has finite-dimensional morphism spaces. Thus, Proposition A.1 gives us:

Definition 3.1. *There is a unique $\mathbf{Q}$-linear functor $L : {}_\mathsf{W}\mathsf{Rep} \to \mathsf{P}_\mathsf{G}(\mathcal{U})$ such that $L(\mathbf{Q}[\mathsf{W}]) = \mathcal{S}$ compatibly with the right actions of W on both sides; we call this $L = \mathcal{S} \otimes_\mathsf{W} -$ the **Springer functor**.*

Our goal will now be to prove Theorem 1.2, i.e., that the Springer functor identifies ${}_{\mathsf{W}}\mathsf{Rep}$ with a Serre subcategory of $\mathsf{P}_{\mathsf{G}}(\mathcal{U})$. We will use Proposition A.3, which says that we can equivalently check the following three properties of the sheaf $\mathcal{S}$ and its associated action of W:

(1) The action map gives an isomorphism $\mathbf{Q}[\mathsf{W}^{\mathsf{op}}] \xrightarrow{\sim} \mathsf{P}_{\mathsf{G}}(\mathcal{S},\mathcal{S})$;
(2) The sheaf $\mathcal{S}$ is semi-simple;
(3) We have $\mathsf{P}^1_{\mathsf{G}}(\mathcal{S},\mathcal{S}) = 0$.

Of these, the second is most quickly taken care of. Indeed, it follows immediately[3] from the decomposition theorem of [BBD] (Theorem 6.2.5), since p is proper and we already know that $\mathcal{S}$ is perverse (Proposition A.3). There are two small subtleties, easily overcome: firstly, the cited theorem only implies that $\mathcal{S}$ is semi-simple in $\mathsf{P}(\mathcal{U})$, not in $\mathsf{P}_{\mathsf{G}}(\mathcal{U})$; but by Proposition B.5, this is the same thing. Secondly, the cited theorem concerns sheaves of $\mathbf{C}$-vector spaces, not sheaves of $\mathbf{Q}$-vector spaces. However, one can go from $\mathbf{C}$ to $\overline{\mathbf{Q}}$ using the Lefschetz principle, and from there to $\mathbf{Q}$ using Galois theory (recall that the decomposition theorem, in the end, is just a statement about certain maps in $\mathsf{D}(\mathcal{X})$ splitting).

Claims (1) and (3), are quite similar, and can in fact be handled simultaneously, using the equivariant derived category D_{G} of Bernstein-Lunts (see Appendix B.5). Since the same same technique will also yield (2) of Theorem 1.3, concerning functoriality of the Springer functor with respect to Levi factors L of G, we give the argument in its general form. But first we must define the restriction and induction functors from Levi factors.

3.2. **Restriction and Induction.** Let G be a connected reductive group, P a parabolic subgroup containing B, and $\mathsf{L} = \mathsf{P}/\mathsf{P}_{\mathsf{u}}$ the Levi factor of P, which we simultaneously think of as the unique Levi subgroup containing T (see [S] Corollary 8.4.4). The reader is encouraged to keep the case $\mathsf{P} = \mathsf{B}$ in mind, when $\mathsf{L} = \mathsf{T}$; this is what will eventually give us (1) and (3) above. Consider the diagram

$$\mathsf{L} \xleftarrow{s} \mathsf{P} \xrightarrow{i} \mathsf{G}$$

Here i is the inclusion, a closed immersion, and s is the (smooth) quotient by the unipotent radical of P. We give G the conjugation action by itself, and L and P the conjugation action by P.

In defining restriction and induction, we will be using the equivariant derived D_{G} formalism of Bernstein-Lunts ([BL]). Even though almost all the sheaves we work with are equivariant *perverse* (even better—see Proposition B.5—equivariant perverse for the action of a connected group), we can't simply use the P_{G}: the restriction and induction functors are multi-step, and intermediate stages are not necessarily perverse.

We recall the forgetful functor $R^{\mathsf{G}}_{\mathsf{P}} : \mathsf{D}_{\mathsf{G}}(\mathsf{G}) \to \mathsf{D}_{\mathsf{P}}(\mathsf{G})$ and its right adjoint $I^{\mathsf{G}}_{\mathsf{P}}$ from Appendix B.5. Here is the first of the two functors.

Definition 3.2. *Define the functor* $\mathrm{Res}^{\mathsf{G}}_{\mathsf{L}} : \mathsf{D}_{\mathsf{G}}(\mathsf{G}) \to \mathsf{D}_{\mathsf{P}}(\mathsf{L})$ *by*

$$\mathrm{Res}^{\mathsf{G}}_{\mathsf{L}}(\mathcal{F}) = s_! i^* R^{\mathsf{G}}_{\mathsf{P}} \mathcal{F}.$$

By definition, $\mathrm{Res}^{\mathsf{G}}_{\mathsf{L}}$ has a right adjoint:

[3] While it only takes us a few sentences to make this argument here, it's worth keeping in mind that the decomposition theorem is the culmination of the whole book [BBD], goes through the Weil conjectures, and, needless to say, is difficult.

Definition 3.3. *Define the functor* $\mathrm{Ind}_{\mathsf{L}}^{\mathsf{G}} : \mathsf{D}_{\mathsf{P}}(\mathsf{L}) \to \mathsf{D}_{\mathsf{G}}(\mathsf{G})$ *by the following formula:*

$$\mathrm{Ind}_{\mathsf{L}}^{\mathsf{G}}(\mathcal{G}) = I_{\mathsf{P}}^{\mathsf{G}} i_* s^!(\mathcal{G}).$$

What is not obvious from the definitions, but nonetheless true, is that these functors often preserve perversity in cases concerning the Springer sheaf and its relatives (see Theorem 1.3 for a more precise statement); this, together with the adjointness of Res and Ind, is what will give us the refined information necessary for proving (1) and (3) above.

We start with Res. What follows is the most important calculation in the whole paper, by a long shot. It is the key to (1) and (3) above, as well as (2) of Theorem 1.3.

Proposition 3.4. *Let* G *be a connected reductive group, and* P *a parabolic subgroup with Levi factor* L*. Then*

$$\mathrm{Res}_{\mathsf{L}}^{\mathsf{G}} \mathcal{S}_{\mathsf{G}} = \mathrm{Ind}_{\mathsf{W}_{\mathsf{L}}}^{\mathsf{W}_{\mathsf{G}}} \mathcal{S}_{\mathsf{L}},$$

the right actions by W_{G} *agreeing on both sides.*

Note: on the right-hand side, what we mean is $\mathcal{S}_{\mathsf{L}} \otimes_{\mathsf{W}_{\mathsf{L}}} \mathbf{Q}[\mathsf{W}_{\mathsf{G}}]$, in the notation of Appendix A; it is simply the obvious analog of the usual induction of representations, and may be defined in the same way, as a direct sum over right cosets $\mathsf{W}_{\mathsf{L}} \backslash \mathsf{W}_{\mathsf{G}}$ with the appropriately-defined action of W_{G}.

Proof. We first note that, since $\mathcal{S}_{\mathsf{G}}$ and $\mathrm{Ind}_{\mathsf{W}_{\mathsf{L}}}^{\mathsf{W}_{\mathsf{G}}} \mathcal{S}_{\mathsf{L}}$ are perverse, we may, by virtue of Proposition B.16, work in the non-equivariant context (and pretend there's no $R_{\mathsf{L}}^{\mathsf{G}}$ in the definition of $\mathrm{Res}_{\mathsf{L}}^{\mathsf{G}}$, so that the left-hand side is just $s_! i^* \mathcal{S}_{\mathsf{G}}$).

Now we outline the proof. Not surprisingly, it follows a similar strategy to the proof of Corollary 2.9, where we constructed the action of W on $\mathcal{S}$. First we will show that the sheaf $s_! i^* \overline{\mathcal{S}_{\mathsf{G}}} \in \mathsf{D}(\mathsf{L})$ is perverse, and moreover equal to its Goresky-Macpherson extension from the regular semi-simple locus $\mathcal{R}_{\mathsf{L}} \subseteq \mathsf{L}$ (see Proposition 2.6). Second, we note that $s_! i^*$ commutes with arbitrary restrictions, so on the one hand it is easy to directly compute this restriction to $\mathcal{R}_{\mathsf{L}}$, and on the other hand (restricting to the unipotent locus) the resulting information on $\overline{\mathcal{S}}$ gives us all we need to know about $\mathcal{S}$.

Now we get down to work. Let $\mathcal{Y} = \mathsf{P} \times_{\mathsf{G}} \widetilde{\mathsf{G}}$ (so that when $\mathsf{P} = \mathsf{B}$, this agrees with the earlier notation in Proposition 2.6). Then by proper base-change, we have

$$s_! i^* \overline{\mathcal{S}_{\mathsf{G}}} = f_! \mathbf{Q}[\dim(\mathsf{G})],$$

with $f : \mathcal{Y} \to \mathsf{P} \to \mathsf{L}$. On the other hand, the Bruhat stratification $\mathcal{B} = \coprod_{w \in \mathsf{W}_{\mathsf{G}}/\mathsf{W}_{\mathsf{L}}} \mathsf{P} \cdot (\dot{w}^{-1}\mathsf{B})$ ([S] Exercise 8.4.6(3)) pulls back to a stratification

$$\mathcal{Y} = \coprod_{w \in \mathsf{W}_{\mathsf{G}}/\mathsf{W}_{\mathsf{L}}} \mathcal{Y}(w)$$

with $\mathcal{Y}(w) \simeq \mathsf{P} \times^{\mathsf{P}^w} \mathsf{P}^w$, just as in Proposition 2.2 (recall $\mathsf{P}^w = \mathsf{P} \cap \dot{w}^{-1}\mathsf{B}\dot{w}$); repeated applications of the standard gluing triangle

$$(j_! j^* \mathcal{F}, \mathcal{F}, i_! i^* \mathcal{F})$$

to the sheaf $\mathcal{F} = \mathbf{Q}[\dim(\mathsf{G})]$ along this stratification then show that $f_! \mathbf{Q}[\dim(\mathsf{G})]$ is gotten by successive extensions of the $f(w)_! \mathbf{Q}[\dim(\mathsf{G})]$, with $f(w) : \mathcal{Y}(w) \hookrightarrow \mathcal{Y} \xrightarrow{f} \mathsf{L}$.

On the other hand, each $f(w)$ factors through the Springer map $p : \widetilde{\mathsf{L}} \to \mathsf{L}$. We give two descriptions of this factoring $\mathcal{Y}(w) \to \widetilde{\mathsf{L}}$—the first being independent of choices, and the second, the one we will work with, being more amenable to calculation. For the first, we use the interpretation of $\widetilde{\mathsf{G}}$ as the set of pairs (x, B) consisting of an element $x \in \mathsf{G}$ and a Borel subgroup B containing it; see Proposition 2.2 (2). Here, the factoring is easily described as a map even from $\mathcal{Y}$ to L: namely, given a pair (x, B) with $x \in \mathsf{P} \cap B$, we have that $(\mathsf{P} \cap B)/\mathsf{P}_\mathsf{U}$ is a Borel subgroup of L, and we send $(x, B) \mapsto (x\mathsf{P}_\mathsf{U}, (\mathsf{P} \cap B)/\mathsf{P}_\mathsf{U})$.

For the second description, we use the Bruhat Decomposition, i.e., we replace the P in the above Bruhat stratification with the more refined U_w (see [S] Corollary 8.3.9); this gives $\mathcal{Y}(w) \simeq \mathsf{U}_w \times \mathsf{P}^w$. Note that, here, we have *chosen* a representative of a class in $\mathsf{W}_\mathsf{G}/\mathsf{W}_\mathsf{L}$, and this isomorphism depends on it (even the target of the isomorphism depends on it). In these terms, the factorization is $\mathsf{U}_w \times \mathsf{P}^w \to \mathsf{L} \times^{\mathsf{B}/\mathsf{P}_\mathsf{U}} \mathsf{B}/\mathsf{P}_\mathsf{U} = \widetilde{\mathsf{L}}$ given by reducing mod U_P on both factors. It's easy to see that these two descriptions of the factoring agree.

Returning to the proof, from the second description we see that the factoring is locally principal with fibers affine space of dimension $\dim(\mathsf{U}_\mathsf{P})$; thus, proper push-forward of the constant sheaf along it merely shifts by $-2\dim(\mathsf{U}_\mathsf{P}) = \dim(\mathsf{L}) - \dim(\mathsf{G})$. Then from the above we deduce that $s_! i^* \overline{\mathcal{S}_\mathsf{G}} = f_! \mathbf{Q}[\dim(\mathsf{G})]$ is gotten by successive extensions of $p_! \mathbf{Q}[\dim(\mathsf{L})] = \mathcal{S}_\mathsf{L}$, and is thus perverse and equal to its Goresky-Macpherson extension from $\mathcal{R}_\mathsf{L}$, by the corresponding fact for $\mathcal{S}_\mathsf{L}$ (see the proof of Corollary 2.9).

This concludes the first step of the above outline; now we see what happens over the étale locus $\mathcal{R}$. There the stratification $\mathcal{Y} = \coprod \mathcal{Y}(w)$ turns into a decomposition into connected components (c.f. Proposition 2.6 (3)), so all the above-discussed extensions are in fact split; thus it's just direct sums of constant sheaves all the way down the calculation until the last step, and we do get an identification

$$s_! i^* \mathcal{S}^\circ_\mathsf{G} = \operatorname{Ind}_{\mathsf{W}_\mathsf{L}}^{\mathsf{W}_\mathsf{G}} \mathcal{S}^\circ_\mathsf{L},$$

at least as sheaves. But checking that the actions agree as well is a simple matter: we just need to see how W_G acts geometrically at every stage of the above argument (recall the definition of the action, Corollaries 2.8 and 2.9). Consider the action of W_G on $\mathcal{Y}$. In terms of the identification of $\mathcal{Y}$ as a closed subvariety of $\mathsf{P} \times \mathcal{B}$, it is just $(x, g\mathsf{B}) \mapsto (x, g\dot{w}^{-1}\mathsf{B})$ (see the fourth paragraph of the proof of Proposition 2.2); so if $\{w_\alpha\}$ stands for a set of representatives of $\mathsf{W}_\mathsf{G}/\mathsf{W}_\mathsf{L}$, then on the component $\mathcal{Y}(w_\alpha)$, the action by w is $(x, u\dot{w_\alpha}^{-1}\mathsf{B}) \mapsto (x, u\dot{w_\alpha}^{-1}\dot{w}^{-1}\mathsf{B})$; rewriting $ww_\alpha = w_\beta w'$ with $w' \in \mathsf{W}_\mathsf{L}$ (using the fact that the α are a set of representatives), we see that this goes to $\mathcal{Y}(w_\beta)$, and there, under the map $\mathcal{Y}(w_\beta) \to \widetilde{\mathsf{L}}$, acts by w'; this is exactly the description of the left induced representation, and when we pass from spaces to functions on the space, we get the right induced representation.

So we have the desired fact for $\mathcal{S}^\circ$; however, by the observations in the outline, this is enough: it Goresky-Macpherson extends to the corresponding statement about $\overline{\mathcal{S}}$, then restricts to the statement we're trying to prove. $\square$

Now let's see how the Springer sheaves interact with $\operatorname{Ind}_\mathsf{L}^\mathsf{G}$. Before beginning, though, note that we can unravel the definition of $\operatorname{Ind}_\mathsf{L}^\mathsf{G} = I_\mathsf{P}^\mathsf{G} i_* s^*(\mathcal{G})$ to some extent. Indeed, the functor $I_\mathsf{P}^\mathsf{G} i_*[\dim(\mathsf{G}/\mathsf{P})]$ (recall from Appendix B.5) also has the following interpretation, by virtue of proper and smooth base-change: given $\mathcal{F} \in \mathsf{D}_\mathsf{P}(\mathsf{P})$, take the unique $\mathcal{F}' \in \mathsf{D}_\mathsf{G}(\mathsf{G} \times_\mathsf{P} \mathsf{P})$ which agrees with $\mathcal{F}$ on bolded pull-back to $\mathsf{G} \times \mathsf{P}$; then

$$i_\mathsf{P}^\mathsf{G} i_* \mathcal{F}[\dim(\mathsf{G}/\mathsf{P})] = a_* \mathcal{F}',$$

$a : \mathsf{D}_\mathsf{G}(\mathsf{G} \times_\mathsf{P} \mathsf{P}) \to \mathsf{D}_\mathsf{G}(\mathsf{G})$ being induced by the action map. Thus

$$\operatorname{Ind}_\mathsf{L}^\mathsf{G}(\mathcal{G}) = a_*(\mathbf{s}^*\mathcal{G})'.$$

This description has two simple consequences:

Lemma 3.5. *Take* $\mathsf{P} = \mathsf{B}$*, so* $\mathsf{L} = \mathsf{T}$*. We have:*

(1) $\overline{\mathcal{S}_{\mathsf{G}}} = \mathrm{Ind}_{\mathsf{T}}^{\mathsf{G}} \mathbf{Q}[\dim(\mathsf{T})]$*;*

(2) $\mathcal{S}_{\mathsf{G}} = \mathrm{Ind}_{\mathsf{T}}^{\mathsf{G}} \mathbf{Q}_e$ *(skyscraper at the identity* $e \in \mathsf{T}$*).*

Proof. The first point is immediate from the above discussion and the definition of $\overline{\mathcal{S}}$ (Definition 2.4—though if you forget this, you're in deep water); the second one then follows by restricting to the unipotent locus at every stage of Ind. $\square$

But in fact, we can make a more general claim:

Proposition 3.6. *Let* G *be a connected reductive group, and* P *a parabolic subgroup with Levi factor* L*. Then*

$$\mathrm{Ind}_{\mathsf{L}}^{\mathsf{G}} \mathcal{S}_{\mathsf{L}} = \mathrm{Res}_{\mathsf{W}_{\mathsf{L}}}^{\mathsf{W}_{\mathsf{G}}} \mathcal{S}_{\mathsf{G}},$$

the right actions by W_{L} *agreeing on both sides (otherwise we wouldn't really have bothered writing* $\mathrm{Res}_{\mathsf{W}_{\mathsf{L}}}^{\mathsf{W}_{\mathsf{G}}}$*).*

Proof. The proof is much simpler than that of Proposition 3.4, but follows the same outline: we first show that the left-hand side is perverse and equal to its Goresky-Macpherson extension from $\mathcal{R}_{\mathsf{G}}$, then we compare on $\mathcal{R}_{\mathsf{G}}$.

In fact, for the first step, we will identify the left-hand side with the right-hand side (as sheaves) a priori. For this, we note that Ind is transitive with respect to inclusions of parabolic subgroups. Indeed, the corresponding fact for Res is simple to verify; then one can invoke adjointness. Given this transitivity, the claim follows from the Lemma.

So now we turn to the regular semi-simple locus. Recall that there is an identification $\widetilde{\mathcal{R}_{\mathsf{G}}} \simeq \mathsf{G}/\mathsf{T} \times \mathcal{R}_{\mathsf{T}}$ under which the Springer map is conjugation and the Weyl group action is $(g\mathsf{T}, t) \mapsto (g\dot{w}^{-1}\mathsf{T}, \dot{w}t\dot{w}^{-1})$ (see the fourth paragraph of the proof of Proposition 2.2), and of course similarly for L; through these identifications, the restriction of the left-hand side to $\mathcal{R}_{\mathsf{G}}$ is just the pushforward along $\mathsf{G} \times^{\mathsf{P}} (\mathsf{P}/\mathsf{T} \times \mathcal{R}_{\mathsf{T}}) \to \mathcal{R}_{\mathsf{G}}$ of the constant sheaf $\mathbf{Q}[\dim(\mathsf{G})]$; however, it's clear that $\mathsf{G} \times^{\mathsf{P}} (\mathsf{P}/\mathsf{T} \times \mathcal{R}_{\mathsf{T}}) \simeq \mathsf{G}/\mathsf{T} \times \mathcal{R}_{\mathsf{T}}$ in the obvious way (the action of P on $\mathsf{P}/\mathsf{T} \times \mathcal{R}_{\mathsf{T}}$ is just left-translation on the left); this lets us conclude. $\square$

3.3. **Back to the proof of Theorem 1.2.** We now finish showing that the Springer functor embeds ${}_{\mathsf{W}}\mathsf{Rep}$ as a Serre subcategory of $\mathsf{P}_{\mathsf{G}}(\mathcal{U})$. Recall that we need both that the action map gives an isomorphism $\mathbf{Q}[\mathsf{W}^{\mathsf{op}}] \xrightarrow{\sim} \mathsf{P}_{\mathsf{G}}(\mathcal{S}, \mathcal{S})$, and that $\mathsf{P}^1_{\mathsf{G}}(\mathcal{S}, \mathcal{S}) = 0$. Let's investigate both claims simultaneously: let $n \in \mathbf{Z}$. We have (hang on, the justifications will come at the end)

$$\begin{aligned} \mathsf{D}_{\mathsf{G}}(\mathcal{S}, \mathcal{S}[n]) &= \mathsf{D}_{\mathsf{G}}(\mathcal{S}, \mathrm{Ind}_{\mathsf{T}}^{\mathsf{G}} \mathbf{Q}_e[n]) \\ &= \mathsf{D}_{\mathsf{B}}(\mathrm{Res}_{\mathsf{T}}^{\mathsf{G}} \mathcal{S}, \mathbf{Q}_e[n]) \\ &= \mathsf{D}_{\mathsf{B}}(\mathrm{Ind}_1^{\mathsf{W}} \mathcal{S}_{\mathsf{T}}, \mathbf{Q}_e[n]) \\ &= \mathsf{D}_{\mathsf{B}}(\mathbf{Q}_e[\mathsf{W}], \mathbf{Q}_e[n]), \end{aligned}$$

the first step by Lemma 3.5, the second by adjointness, the third by Proposition 3.4, and the fourth by... well, I'll leave the fourth to the reader.

If we're only considering $n \leq 1$, then by Lemma B.7, the above equality can also be written

$$\mathsf{P}^n_{\mathsf{G}}(\mathcal{S}, \mathcal{S}) = \mathsf{P}^n_{\mathsf{B}}(\mathbf{Q}_e[\mathsf{W}], \mathbf{Q}_e).$$

But this last morphism space is over a point; and, B being connected, the category $\mathsf{P}_{\mathsf{B}}(\bullet)$ is just vector spaces (c.f. Lemma B.12); so we conclude

$$\mathsf{P}^n_{\mathsf{G}}(\mathcal{S},\mathcal{S}) = \mathsf{Vect}^n_{\mathsf{Q}}(\mathbf{Q}[\mathsf{W}],\mathbf{Q})$$

for $n \leq 1$.

Taking $n = 1$ immediately gives $\mathsf{P}^1_{\mathsf{G}}(\mathcal{S},\mathcal{S}) = 0$, one of our desired facts. What remains is to prove that the action map gives an isomorphism $\mathbf{Q}[\mathsf{W}^{\mathsf{op}}] \xrightarrow{\sim} \mathsf{P}_{\mathsf{G}}(\mathcal{S},\mathcal{S})$. For this we take $n = 0$ above. By the claim about the actions agreeing in Proposition 3.4, we then see that there is an isomorphism of (left) W-representations $\mathsf{P}_{\mathsf{G}}(\mathcal{S},\mathcal{S}) \simeq \mathbf{Q}[\mathsf{W}]$. To conclude, it suffices to use the following lemma:

Lemma 3.7. *Let k be a field, A a finite k-algebra, and $\varphi : A \to B$ a map of k-algebras; in particular, B becomes a right A-module. If B is abstractly isomorphic to A as a right A-module, then φ is an isomorphism.*

Proof. The right-A-module A has a generator; since $B \cong A$, so does B, so say $b \cdot A = B$. Then the composition

$$A \xrightarrow{\varphi} B \xrightarrow{b} B$$

is surjective, the right map being left-multiplication by b; but all vector spaces involved have the same dimension, so all maps must be isomorphisms. □

4. Complements

4.1. **The Springer correspondence.** We have finished the proof of Theorem 1.2, that the Springer functor gives a Serre embedding of ${}_{\mathsf{W}}\mathsf{Rep}$ into $\mathsf{P}_{\mathsf{G}}(\mathcal{U})$. Now we give some consequences and supplementary facts.

Firstly, we note that since $\mathsf{P}_{\mathsf{G}}(\mathcal{U})$ is artinian and noetherian (Corollary B.6), its Serre subcategories are in one-to-one correspondence with subsets of S, the set of isomorphism classes of its simple objects; on the other hand, Proposition B.13 identifies S with the set of pairs $(\mathcal{O},\tau)$ consisting of a unipotent conjugacy class $\mathcal{O}$ and an irreducible finite-dimensional representation τ of the group $\mathsf{Z}(u)/\mathsf{Z}(u)^\circ$ of components of the centralizer of a fixed $u \in \mathcal{O}$.

Thus, we deduce:

Proposition 4.1. *The Springer functor induces a bijection $\rho \mapsto (\mathcal{O}_\rho, \tau_\rho)$, called the **Springer correspondence**, between the set of irreducible representations of W and some subset R of S.*

It's reasonable to ask for information about this subset and the induced bijection. We will soon see that, in some sense, the subset R is quite large; to prove this (and of course also because of its independent interest), we will give Springer's description of his correspondence ([S2]), a.k.a. finding representations in the cohomology groups.

Theorem 4.2. *Let $\mathcal{O}$ be a unipotent conjugacy class, and $u \in \mathcal{O}$. Then*

$$H^{2\dim(\mathcal{B}_u)}(\mathcal{B}_u;\mathbf{Q}) = \bigoplus_{\rho:\mathcal{O}_\rho=\mathcal{O}} \tau_\rho \otimes \rho^\vee,$$

the sum being over all irreducible representations ρ of W whose corresponding conjugacy class equals $\mathcal{O}$.

Proof. Applying the Springer functor L to the decomposition

$$\mathbf{Q}[\mathsf{W}] = \bigoplus_{\rho} \rho \otimes \rho^{\vee}$$

of the regular representation into left-right $\mathsf{W}-\mathsf{W}$ modules (the sum being over all irreducible ρ), we deduce that, as a sheaf with a right W-action,

$$\mathcal{S} = \bigoplus_{\rho} L(\rho) \otimes \rho^{\vee}.$$

On the other hand, Proposition B.13 realizes $L(\rho)$ as the Goresky-Macpherson extension $j_{!*}\mathcal{L}_\rho[\dim(\mathcal{O}_\rho)]$ of the local system $\mathcal{L}_\rho$ corresponding to τ_ρ along the inclusion $j : \mathcal{O}_\rho \to \mathcal{U}$; we can use this to pick out the ρ component, or at least almost.

Indeed, if we restrict $\mathcal{S}$ to $\mathcal{O}$, we will kill off all the ρ components except those with $\mathcal{O} \subseteq \overline{\mathcal{O}_\rho}$; then if we further take cohomology in degree $-\dim(\mathcal{O})$, we'll kill all except those with $\mathcal{O}_\rho = \mathcal{O}$, by the basic property of Goresky-Macpherson extension ([BBD] Proposition 2.1.9). We deduce that for $u \in \mathcal{O}$,

$$H^{-\dim(\mathcal{O})}(\mathcal{S}|_u) = \bigoplus_{\rho : \mathcal{O}_\rho = \mathcal{O}} \tau_\rho \otimes \rho^{\vee}.$$

On the other hand, by proper base-change and the definition of $\mathcal{S}$, this translates to the desired

$$H^{2\dim(\mathcal{B}_u)}(\mathcal{B}_u; \mathbf{Q}) = \bigoplus_{\rho : \mathcal{O}_\rho = \mathcal{O}} \tau_\rho \otimes \rho^{\vee},$$

where we have also used the equality of dimensions in Proposition 2.2 (2). □

Now we can see that R is big.

Corollary 4.3. *For each conjugacy class $\mathcal{O}$, there is an element of R whose first component is $\mathcal{O}$.*

Proof. This follows immediately, since $H^{2\dim(\mathcal{B}_u)}(\mathcal{B}_u; \mathbf{Q})$ is certainly nonzero; in fact it has basis the components of $\mathcal{B}_u$ (recall that $\mathcal{B}_u$ is a projective variety: Proposition 2.2 (3)). □

Another natural question is where the trivial representation of W goes under the Springer correspondence. If someone tries to tell you it goes to something other than the identity conjugacy class and the open conjugacy class, you should contact the nearest psychiatric clinic. In fact,

Proposition 4.4. *Under the Springer correspondence, the trivial representation $1 \in {}_{\mathsf{W}}\mathsf{Rep}$ goes to the pair $(\mathcal{O}, 1)$ where $\mathcal{O}$ is the open orbit in $\mathcal{U}$.*

Proof. Recall that the action of W on $\mathcal{S}$ was obtained functorially from the action on $\mathcal{S}^\circ$, through Goresky-Macpherson extension followed by restriction (Corollary 2.9). From Proposition A.2 (1), we see that it suffices to see where $\mathcal{S}^\circ \otimes_{\mathsf{W}} 1$ goes under the same extension-restriction procedure.

But from the definition of the action on $\mathcal{S}^\circ$, it's clear that $\mathcal{S}^\circ \otimes_{\mathsf{W}} 1 = (\mathcal{S}^\circ)^{\mathsf{W}}$ is just the constant sheaf $\mathbf{Q}[\dim(\mathsf{G})]$ on $\mathcal{R}$. If we Goresky-Macpherson extend this in a leisurely way, stopping for a rest at the (open and G-stable, by properness and G-equivariance) locus $\mathcal{Z}_0 \subseteq \mathsf{G}$ where $\overline{p}$ has finite fibers, we see that $\mathcal{S}^{\mathsf{W}}|_{\mathcal{O}} = \mathbf{Q}[\dim(\mathcal{U})]$, making the only possibility $1 \mapsto (\mathcal{O}, 1)$. □

4.2. **Intertwining.** Now we will give the proof of Theorem 1.3, concerning what the Springer functor L does to certain usual operations on representations. We state it again for convenience:

Theorem 4.5. *Let G be a connected reductive group, and L the Springer functor.*

(1) L intertwines duality on representations and Verdier duality D on sheaves;

(2) Let P be a parabolic subgroup of G with Levi factor L, giving an inclusion $\mathsf{W}_\mathsf{L} \subseteq \mathsf{W}_\mathsf{G}$ of Weyl groups. Then:

(a) L intertwines $\mathrm{Res}_{\mathsf{W}_\mathsf{L}}^{\mathsf{W}_\mathsf{G}}$ and the above $\mathrm{Res}_\mathsf{L}^\mathsf{G}$;

(b) L intertwines $\mathrm{Ind}_{\mathsf{W}_\mathsf{L}}^{\mathsf{W}_\mathsf{G}}$ and the above $\mathrm{Ind}_\mathsf{L}^\mathsf{G}$.

(3) Let G' be another connected reductive group, with uninpotent locus $\mathcal{U}'$ and Weyl group W'. Then $\mathsf{G}\times\mathsf{G}'$ has unipotent locus $\mathcal{U}\times\mathcal{U}'$ and Weyl group $\mathsf{W}\times\mathsf{W}'$, and L intertwines the external product $\boxtimes$ on representations and the external product $\boxtimes$ on perverse sheaves.

Proof. By Proposition A.2, we need to see that:

(I) $D(\mathcal{S}) = \mathcal{S}$, the left action of W agreeing on both sides (on the right side, it comes from applying group inversion to the right action);

(II) $\mathrm{Res}_\mathsf{L}^\mathsf{G} \mathcal{S}_\mathsf{G} = \mathrm{Ind}_{\mathsf{W}_\mathsf{L}}^{\mathsf{W}_\mathsf{G}} \mathcal{S}_\mathsf{L}$ and $\mathrm{Ind}_\mathsf{L}^\mathsf{G} \mathcal{S}_\mathsf{L} = \mathrm{Res}_{\mathsf{W}_\mathsf{L}}^{\mathsf{W}_\mathsf{G}} \mathcal{S}_\mathsf{G}$, the right actions agreeing on both sides in both cases;

(III) $\mathcal{S}_{\mathsf{G}\times\mathsf{G}'} = \mathcal{S}_\mathsf{G} \boxtimes \mathcal{S}_{\mathsf{G}'}$, the actions of $\mathsf{W}\times\mathsf{W}'$ agreeing on both sides.

Actually, there is a small problem in applying Proposition A.2 to (II) in order to deduce (2) of the Theorem, coming from the fact that we don't know a priori that $\mathrm{Res}_\mathsf{L}^\mathsf{G} \circ L_\mathsf{G}$ and $\mathrm{Ind}_\mathsf{L}^\mathsf{G} \circ L_\mathsf{L}$ land inside the abelian categories of perverse sheaves, but only the additive derived categories. However, the *uniqueness* part of Proposition A.1 works just as well for additive categories as abelian ones, as the proof shows, and we only need the uniqueness here. So the reduction does work, and we can proceed to the proofs of (I), (II), and (III).

Statement (II) says the same thing as Propositions 3.4 and 3.6. For statement (I), we follow the same outline as in the proof of those Propositions: we already noted in Proposition 2.5 and Corollary 2.9 that the sheaves $\overline{\mathcal{S}}$ and $\mathcal{S}$ are self-dual, so it's just a matter of making sure the actions agree, and by functoriality we can just check this for $\mathcal{S}^\circ$ on the regular semi-simple locus; there we can simply recall that Verdier duality commutes with usual duality under the correspondence between local systems and representations of the fundamental group. Statement (III) is simple, using the same outline as always; we leave it for the interested reader. □

4.3. **The case of GL_n.** We specialize the Springer theory to the nicest case: $\mathsf{G} = \mathsf{GL}_\mathsf{n}$. The reason this case is so nice is the following:

Lemma 4.6. *In GL_n, all centralizers are connected.*

Proof. Let $A \in \mathsf{GL}_\mathsf{n}$. We show that any two elements $X, Y \in \mathsf{Z}_{\mathsf{GL}_\mathsf{n}}(A)$ are joined by a curve in $\mathsf{Z}_{\mathsf{GL}_\mathsf{n}}(A)$. Indeed, if t runs through A^1, all of the matrices

$$tX + (1-t)Y,$$

centralize A, and only finitely many of them (a proper Zariski-closed subset) can have determinant zero. □

Thus our subset S of pairs $(\mathcal{O}, \tau)$ (see Section 3.4) is just the set of unipotent conjugacy classes; then Corollary 3.10 lets us deduce simultaneously the two claims of the following theorem:

Theorem 4.7. *For $\mathsf{G} = \mathsf{GL_n}$, the Springer functor is an equivalence of categories, and the Springer correspondence is a bijection between irreducible representations of $\mathsf{S_n}$ and unipotent conjugacy classes in $\mathsf{GL_n}$.*

We note that this bijection was proved without any counting (unlike, for instance, the proof of the classical bijection in [FH] Section 4.2, which only shows injectivity, then uses counting to conclude), and is thus fairly nice from a combinatorial perspective—except, perhaps, for the fact that the (absolutely crucial) action of W on $\mathcal{S}$ had a non-explicit construction: it involved Goresky-Macpherson extension followed by restriction to a complementary subspace. All the rest is explicit, however: see Proposition A.1.

Springer's interpretation ([S2]) of the inverse map of the Springer correspondence also becomes very nice in the case of $\mathsf{GL_n}$: for a unipotent conjugacy class $\mathcal{O}$ and $u \in \mathcal{O}$, the corresponding irreducible representation of $\mathsf{S_n}$ is just the cohomology group

$$H^{2\dim(\mathcal{B}_u)}(\mathcal{B}_u; \mathbf{Q}).$$

Appendix A. Representation theory over k-linear abelian categories

Our main object of interest, the Springer sheaf $\mathcal{S}$, lives in a certain abelian category (of perverse sheaves), and carries moreover an action of a finite group W (the Weyl group of a reductive group). This section, quite formal, explores some operations which arise in such a context.

Let k be a field, and W a finite group. Consider a k-linear abelian category P, and suppose that all of the morphism spaces of P are finite-dimensional. We will be entertaining Morita-like discussions of the left adjoint k-linear functors from the category ${}_{\mathsf{W}}\mathsf{Rep}$ of finite-dimensional k-representations of W to P. But first, notation: let $\mathsf{P_W}$ be the (k-linear abelian) category whose objects are the objects $\mathcal{S} \in \mathsf{P}$ with a right action of W, and whose morphisms are the W-equivariant morphisms in P. The basic proposition is:

Proposition A.1. *Let P be a k-linear abelian category with finite-dimensional morphism spaces. Then the category of left adjoint functors ${}_{\mathsf{W}}\mathsf{Rep} \to \mathsf{P}$ (morphisms being natural transformations) is equivalent to $\mathsf{P_W}$, via evaluation of the functor at the element $k[\mathsf{W}]$ (which carries the obvious right action).*

Also, if the order of W is invertible in k, every k-linear functor ${}_{\mathsf{W}}\mathsf{Rep} \to \mathsf{P}$ is a left adjoint; hence, also, we can omit "left adjoint" from the previous sentence.

Proof. In fact, another interpretation of $\mathsf{P_W}$ is as the category of P-valued functors from the full subcategory of ${}_{\mathsf{W}}\mathsf{Rep}$ consisting of just the object $k[\mathsf{W}]$; our association is just restriction of functors. Because every object of ${}_{\mathsf{W}}\mathsf{Rep}$ has a finite presentation by $k[\mathsf{W}]$, and left adjoints preserve right-exact sequences, an extension of an element $\mathcal{S}$ of $\mathsf{P_W}$ to a left adjoint functor L on all of ${}_{\mathsf{W}}\mathsf{Rep}$ is necessarily unique, given by

$$L(\rho) = \operatorname{coker}(\mathcal{S}^m \to \mathcal{S}^n)$$

if $k[\mathsf{W}]^m \to k[\mathsf{W}]^n \to \rho \to 0$ is a presentation of ρ. To finish, we just need to see that this prescription actually defines a left adjoint functor. But indeed, since $k[\mathsf{W}] \in {}_{\mathsf{W}}\mathsf{Rep}$ just co-represents the forgetful functor to vector spaces, L is clearly left adjoint to $\mathsf{P}(\mathcal{S}, -)$; and recall that functoriality then comes for free.

If the order of W is invertible in k, then ${}_{\mathsf{W}}\mathsf{Rep}$ is semi-simple, and so any k-linear functor from it is exact, hence right-exact; then the above argument furnishes a right adjoint. $\square$

For $\mathcal{S} \in \mathsf{P}_{\mathsf{W}}$, denote by $\mathcal{S} \otimes_{\mathsf{W}} -$ the corresponding functor ${}_{\mathsf{W}}\mathsf{Rep} \to \mathsf{P}$. The next proposition gives compatibility properties of the association $\mathcal{S} \mapsto \mathcal{S} \otimes_{\mathsf{W}} -$.

Proposition A.2. *Suppose that the order of* W *is invertible in* k*. Let* P *and* Q *be* k*-linear abelian categories with finite-dimensional morphism spaces, and let* $F : \mathsf{P} \to \mathsf{Q}$ *be a* k*-linear functor between them (not necessarily exact). Then for* $\mathcal{S} \in \mathsf{P}_{\mathsf{W}}$*, we have* $F(\mathcal{S}) \in \mathsf{Q}_{\mathsf{W}}$ *in the obvious way, and for* $\rho \in {}_{\mathsf{W}}\mathsf{Rep}$*,*

$$F(\mathcal{S} \otimes_{\mathsf{W}} \rho) = F(\mathcal{S}) \otimes_{\mathsf{W}} \rho \tag{1}$$

(actually an isomorphism of functors ${}_{\mathsf{W}}\mathsf{Rep} \to \mathsf{Q}$*). On the other side, if* W' *is a subgroup of* W*, then* $\mathcal{S}$ *also has a restricted* W'*-action, and*

$$\mathrm{Res}(\mathcal{S}) \otimes_{\mathsf{W}'} \rho' = \mathcal{S} \otimes_{\mathsf{W}} \mathrm{Ind}(\rho'), \tag{2}$$

both being functors (in ρ'*) from* ${}_{\mathsf{W}'}\mathsf{Rep}$ *to* P*; and if* $\mathcal{S}' \in \mathsf{P}_{\mathsf{W}'}$*, then* $\mathrm{Ind}(\mathcal{S}') := \mathcal{S}' \otimes_{\mathsf{W}'} k[\mathsf{W}]$ *has a right action of* W*, and*

$$\mathrm{Ind}(\mathcal{S}') \otimes_{\mathsf{W}} \rho = \mathcal{S}' \otimes_{W'} \mathrm{Res}(\rho), \tag{3}$$

both being functors from ${}_{\mathsf{W}}\mathsf{Rep}$ *to* P*.*

Proof. In cases respectively (1), (2), (3), by the previous proposition, we need only check that evaluation at respectively $\rho = k[\mathsf{W}]$, $\rho' = k[\mathsf{W}']$, $\rho = k[\mathsf{W}]$ gives isomorphic members of respectively, Q_{W}, $\mathsf{P}_{\mathsf{W}'}$, P_{W}. In all cases, this is trivial. $\square$

Given the equivalence in Proposition A.1, one would expect properties of $\mathcal{S} \otimes_{\mathsf{W}} -$ to be reflected in properties of $\mathcal{S}$. The next proposition gives a few examples of this.

Proposition A.3. *Suppose again that the order of* W *is invertible in* k*. Let* $\mathcal{S} \in \mathsf{P}_{\mathsf{W}}$*, and set* $L = \mathcal{S} \otimes_{\mathsf{W}} -$*. This is automatically an exact functor,* ${}_{\mathsf{W}}\mathsf{Rep}$ *being semi-simple. We furthermore have*

1. *L is fully faithful if and only if the map $k[\mathsf{W}^{\mathsf{op}}] \to \mathsf{P}(\mathcal{S}, \mathcal{S})$ induced by the action of W is an isomorphism;*
2. *Assuming the equivalent conditions of (1), L is thick (i.e. its essential image is closed under subquotients) if and only if $\mathcal{S}$ is semi-simple;*
3. *Assuming the conditions of (1) and (2), L is Serre (i.e. its essential image is also closed under extensions) if and only if $\mathsf{P}^1(\mathcal{S}, \mathcal{S}) = 0$.*

Proof. For (1), note that, since L is a left adjoint, it is fully faithful if and only if its unit $id_{\mathsf{P}} \to R \circ L$ is an isomorphism. By Proposition A.1, we can check this just at the object $k[\mathsf{W}]$, where, recalling the definition of the right adjoint in the proof of that proposition, it becomes exactly the required condition.

For (2), "only if" is easy: a thick fully faithful exact functor sends semi-simple objects to semi-simple objects, and $k[\mathsf{W}] \in {}_{\mathsf{W}}\mathsf{Rep}$ is semi-simple.

For "if", we first note that, since every representation is a direct sum of direct summands of $k[\mathsf{W}]$, everything in the essential image of L is a direct sum of direct summands of $\mathcal{S}$, and is thus semi-simple with simple parts a subset of the simple parts of $\mathcal{S}$. Since a subquotient of a semi-simple object is also semi-simple, with simple parts moreover a subset of those of the original object, it will suffice to show that the simple parts of $\mathcal{S}$ are in the essential image of L. However, a decomposition of $k[\mathsf{W}]$ into simple parts yields a decomposition of $\mathcal{S}$ into parts with trivial endomorphism ring, by fullness of L; and these parts must be semi-simple, since $\mathcal{S}$ is. But then they can't have more then one simple part, or else they'd have too many endomorphisms; thus they are simple. So we have decomposed $\mathcal{S}$ into simple parts in the essential image of L, and this completes the proof of "if".

For (3), recall that $\mathsf{P}^1(\mathcal{S},\mathcal{S})$ classifies extensions of $\mathcal{S}$ by itself. If L is Serre, each such extension comes from an extension in ${}_{\mathsf{W}}\mathsf{Rep}$, all of which are trivial; so $\mathsf{P}^1(\mathcal{S},\mathcal{S}) = 0$. Conversely, if $\mathsf{P}^1(\mathcal{S},\mathcal{S})$ is trivial, then so is $\mathsf{P}^1(\mathcal{T},\mathcal{T}')$ if $\mathcal{T}$ and $\mathcal{T}'$ are direct sums of direct summands of $\mathcal{S}$; but everything in the essential image of L is a direct sum of direct summands of $\mathcal{S}$, since everything in ${}_{\mathsf{W}}\mathsf{Rep}$ is a direct sum of direct summands of $k[\mathsf{W}]$. Thus all the relevant extensions are split, and so trivially remain in the essential image of L. $\square$

We can illustrate this theory with a classical example, which, despite its involving many of our main objects, is not used in the paper.

Example (Weyl's construction). Let V be a finite-dimensional complex vector space, and let P be the category of finite-dimensional representations of the reductive group $\mathsf{GL}(V)$. For all $n \geq 1$, the element $V^{\otimes n}$ of P carries the obvious right action of $\mathsf{S_n}$. If $n \leq \dim(V)$, then the three conditions in the above Proposition are satisfied—the first by [FH] Lemma 6.23 and Exercise 6.29, and the last two by the fact that P is semi-simple ([FH] Exercise 15.51). Thus, we have a Serre embedding of the category of finite-dimensional representations of $\mathsf{S_n}$ into P, and in particular irreducible representations of $\mathsf{S_n}$ give rise to irreducible representations of $\mathsf{GL}(V)$.

If $n > \dim(V)$, the only thing that fails in the above discussion is the injectivity of the action map $\mathbf{C}[\mathsf{S_n^{op}}] \to \mathsf{P}(V^{\otimes n}, V^{\otimes n})$; Exercise 6.29 of [FH] only implies surjectivity. But the proof of the above Proposition shows that, even just assuming surjectivity, irreducible representations of $\mathsf{S_n}$ still give simple elements of P, i.e. irreducible representations of $\mathsf{GL}(V)$ (though many will be trivial: see [FH] Theorem 16.3 (1)). One can show ([FH], Prop. 15.47) that all irreducible representations of $\mathsf{GL}(V)$ are obtained this way, for varying n; this is Weyl's construction.

We also remark that, if A is any finite k-algebra, the results in this section admit an easy extension to a description of the left-adjoint functors from the category of finitely-presented left A-modules to P, with the same proofs (the hypothesis about the order of W should be replaced by the requirement that A be semi-simple, and the analog of Ind is of course the usual base change to an extension ring). We have avoided this generality in order to have our language resemble that of the rest of the paper more closely.

Appendix B. G-equivariant perverse sheaves

In fact, the abelian category in which the Springer sheaf $\mathcal{S}$ most naturally lives is not a usual category of perverse sheaves on a variety, but rather a category of G-*equivariant* perverse sheaves. In this section, we will give the definitions and results necessary to be able to use such a notion.

B.1. **Preliminaries on smooth maps.** The key to the definition of G-equivariant perverse sheaves is that normal perverse sheaves satisfy smooth descent (Proposition B.3). Thus, we start with a study of smooth maps and their interaction with perverse sheaves.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a smooth map of varieties, of relative dimension d (i.e., all fibers are equidimensional of constant dimension $d \geq 0$; in particular f is surjective). By [BBD] page 109, the functor $f^*[d] : \mathsf{D}(\mathcal{Y}) \to \mathsf{D}(\mathcal{X})$ is t-exact, and so induces an exact functor on perverse sheaves, which is moreover fully faithful and thick if the fibers of f are connected ([BBD] Proposition 4.2.5 and Corollary 4.2.6.2). These functors $f^*[d]$ being fundamental for us, we introduce notation and an oft-to-be-cited lemma concerning them.

Definition B.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a smooth map of relative dimension d. We set $\mathbf{f}^*= f^*[d] = f^![-d]$ (note, self-dual), a functor $\mathsf{D}(\mathcal{Y}) \to \mathsf{D}(\mathcal{X})$. If s is a section of f, we similarly use $\mathbf{s}^*$ to denote $s^*[-d]$, a functor $\mathsf{D}(\mathcal{X}) \to \mathsf{D}(\mathcal{Y})$.*

Lemma B.2. *Notation as in the above definition.*

(1) *On the essential image of $\mathbf{f}^*$, we have that $\mathbf{s}^*= s^![d]$ (also self-dual) and is t-exact.*
(2) *Now restrict $\mathbf{f}^*$ to a functor $\mathsf{P}(\mathcal{Y}) \to \mathsf{P}(\mathcal{X})$. If the fibers of f are connected, then $\mathbf{s}^*$ restricted the essential image of $\mathbf{f}^*$ gives the inverse functor to $\mathbf{f}^*$ (c.f.* [BBD] *Proposition 4.2.5).*

Proof. For (1), note that, if D is the Verdier duality operation,

$$D\mathbf{s}^*\mathbf{f}^*\mathcal{F} = D\mathcal{F} = \mathbf{s}^*\mathbf{f}^* D\mathcal{F} = Ds^![d]\mathbf{f}^*\mathcal{F};$$

applying D gives the formula. For t-exactness, we just saw that $\mathbf{s}^*$ is self-dual, so it suffices to see right-t-exactness. Let $\mathcal{F} \in \mathsf{D}(\mathcal{Y})$, and let $\{\mathcal{Y}_i\}$ be a stratification of $\mathcal{Y}$ on which $\mathcal{F}$ has locally constant cohomology. Then $\{f^{-1}(\mathcal{Y}_i)\}$ is such a stratification for $\mathbf{f}^*\mathcal{Y}$, and $\{\mathcal{Y}_i\}$ again is such a stratification for $\mathbf{s}^*\mathbf{f}^*\mathcal{F}$; so the dimension estimate $\dim(\mathcal{Y}_i) + d = \dim(f^{-1}(\mathcal{Y}_i))$ lets us conclude.

Part (2) is obvious just from $\mathbf{s}^*\mathbf{f}^*= id$. ☐

We will soon state and prove smooth descent for perverse sheaves; but first, recall the usual simplicial variety associated to a map $f : \mathcal{X} \to \mathcal{Y}$; here is the first part of it (omitting the degeneracies, which are all base changes of the diagonal $d : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$):

$$\mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \;\substack{\xrightarrow{p_{12}} \\ \xrightarrow{p_{13}} \\ \xrightarrow{p_{23}}}\; \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \;\substack{\xrightarrow{p_{1}} \\ \xrightarrow{p_{2}}}\; \mathcal{X}$$

Note that, when f is smooth, these fiber products are the same in varieties and in schemes: see Lemma 2.3. Now we state the result.

Proposition B.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be smooth of relative dimension d. The functor $\mathbf{f}^*$ induces an equivalence of categories between $\mathsf{P}(\mathcal{Y})$ and the category consisting of pairs $(\mathcal{F}, \iota)$ where $\mathcal{F} \in \mathsf{P}(\mathcal{X})$ and $\iota : \mathbf{p_1}^*\mathcal{F} \simeq \mathbf{p_2}^*\mathcal{F}$ is an isomorphism satisfying the compatibilities*

(1) $\mathbf{d}^*(\iota)= id_{\mathcal{F}}$;
(2) $\mathbf{p_{13}}^*(\iota) = \mathbf{p_{23}}^*(\iota) \circ \mathbf{p_{12}}^*(\iota)$.

(The maps $(\mathcal{F}, \iota) \to (\mathcal{F}', \iota')$ in this category being those maps $\mathcal{F} \to \mathcal{F}'$ which intertwine ι and ι').

Proof. More precisely, the functor is $\mathcal{G} \mapsto (\mathbf{f}^*\mathcal{G}, id)$, where id actually stands for the canonical isomorphism $\mathbf{p_1^*}\mathbf{f}^*\mathcal{G} \simeq \mathbf{p_2^*}\mathbf{f}^*\mathcal{G}$ arising from the equality $f \circ p_1 = f \circ p_2$; note that the compatibilities (1) and (2) are easily verified in this case.

Since perverse sheaves are étale-local ([BBD] 2.1.24) and the whole situation plays nice with base-change, the fact that surjective smooth morphisms have étale-local sections lets us reduce to the case where f itself has a section s. But then we can define an inverse functor, namely $(\mathcal{F}, \iota) \mapsto \mathbf{s}^*\mathcal{F}$ (part of the claim being that this is perverse; we'll check this at the end). It's clear that the composed functor on $\mathsf{P}(\mathcal{Y})$ is the identity; for the other direction, we need that $(\mathcal{F}, \iota) \simeq (\mathbf{f}^*\mathbf{s}^*\mathcal{F}, \mathbf{id})$. But indeed,

$$\mathbf{f}^*\mathbf{s}^*\mathcal{F} = (\mathbf{sf} \times \mathbf{id})^*\mathbf{p_1}^*\mathcal{F} \simeq (\mathbf{sf} \times \mathbf{id})^*\mathbf{p_2}^*\mathcal{F} = \mathcal{F},$$

the last equality coming from $id_{\mathcal{X}} = p_2 \circ (sf \times id)$; and it is easy to see (using condition (2)) that this isomorphism, defined via ι, intertwines ι and id. To finish, we need that $\mathbf{s}^*\mathcal{F}$ is perverse. However, the just-proved isomorphism $\mathcal{F} \simeq \mathbf{f}^*\mathbf{s}^*\mathcal{F}$ shows that $\mathcal{F}$ is in the essential image of $\mathbf{f}^*$, and so this follows from Lemma B.2 (2). □

Now we turn to our subject of interest. Let G be a algebraic group of dimension d, and $\mathcal{X}$ a G-space. We also have a simplicial variety in this situation; here is the first part of it (again, we omit the degeneracies, which are all base-changes of the identity section $e : \bullet \to \mathsf{G}$):

$$\mathsf{G} \times \mathsf{G} \times \mathcal{X} \;\overset{id\times a}{\underset{id\times p}{\overset{m\times id}{\rightrightarrows\!\!\!\to}}}\; \mathsf{G} \times \mathcal{X} \;\underset{p}{\overset{a}{\rightrightarrows}}\; \mathcal{X}$$

Here a is the action map, p is projection, and $m : \mathsf{G} \times \mathsf{G} \to \mathsf{G}$ is the group multiplication. Recall that each of these is smooth with fibers isomorphic to G (in fact, each is isomorphic to a projection from G). Thus, we may apply Lemma B.2 to any of them. Now, here is the main definition:

Definition B.4. *The category $\mathsf{P}_\mathsf{G}(\mathcal{X})$ of G-equivariant perverse sheaves on $\mathcal{X}$ is defined to be the category of pairs $(\mathcal{F}, \iota)$ where $\mathcal{F} \in \mathsf{P}(\mathcal{X})$ and $\iota : \mathbf{a}^*(\mathcal{F}) \simeq \mathbf{p}^*(\mathcal{F})$ is an isomorphism satisfying the compatibilities*

(1) $(\mathbf{e} \times \mathbf{id})^*(\iota) = id_{\mathcal{F}}$;
(2) $(\mathbf{m} \times \mathbf{id})^*(\iota) = (\mathbf{id} \times \mathbf{p})^*(\iota) \circ (\mathbf{id} \times \mathbf{a})^*(\iota)$.

Remark. One obtains an equivalent category if one leaves out condition (1): this is because, given any $\iota : \mathbf{a}^*(\mathcal{F}) \simeq \mathbf{p}^*(\mathcal{F})$, we can replace it with $\iota \circ \mathbf{a}^*(\mathbf{e} \times \mathbf{id})^*(\iota^{-1})$ and obtain an isomorphic object satisfying (1). This remark (which could have also been made about Proposition B.3) will simplify matters when we turn to functoriality (Appendix B.5)—though one can still verify, using Lemma B.2 (1), that the functors we define will preserve condition (1).

The intuition is that we think of $\mathsf{P}_\mathsf{G}(\mathcal{X})$ as being the category of perverse sheaves on the quotient $\mathcal{X}/\mathsf{G}$. It's clear that $\mathsf{P}_\mathsf{G}(\mathcal{X})$ is a $\mathbf{Q}$-linear abelian category admitting an exact, faithful (forgetful) functor to $\mathsf{P}(\mathcal{X})$, which we think of as $\mathbf{q}^*$, with $q : \mathcal{X} \to \mathcal{X}/\mathsf{G}$ the quotient map; c.f. Lemma B.2. This intuition guides us in every case; however, we note that it is more than just intuition in certain cases (take $\mathsf{H} = \mathsf{G}$ in Proposition B.9).

We also want to remark that the notion of a G-equivariant perverse sheaf affords a substantial technical simplification in the case when G is connected (a fact which one easily guesses, thinking of the forgetful functor as $\mathbf{q}^*$ and recalling [BBD] Proposition 4.2.5):

Proposition B.5. *Suppose that* G *is connected. Then the forgetful functor* $\mathsf{P}_{\mathsf{G}}(\mathcal{X}) \to \mathsf{P}(\mathcal{X})$ *is fully faithful and thick, and its essential image consists of those* $\mathcal{F} \in \mathsf{P}(\mathcal{X})$ *for which there merely exists an isomorphism* $\mathbf{a}^*\mathcal{F} \simeq \mathbf{p}^*\mathcal{F}$.

Proof. We first identify the essential image. Suppose we have $\mathcal{F} \in \mathsf{P}(\mathcal{X})$ with $\iota : \mathbf{a}^*\mathcal{F} \simeq \mathbf{p}^*\mathcal{F}$. By the reasoning of the remark following the definition, we can assume ι satisfies (1) of the definition. Then it automatically satisfies (2). Indeed, (2) is a question of equality among two maps $(\mathbf{m} \times \mathbf{id})^*\mathbf{a}^*\mathcal{F} \to (\mathbf{id} \times \mathbf{p})^*\mathbf{p}^*\mathcal{F}$; but both give $id_{\mathcal{F}}$ under $(\mathbf{e} \times \mathbf{id})^*(\mathbf{e} \times \mathbf{id} \times \mathbf{id})^*$, so they must be equal, by Lemma B.2 (2).

Now we show full fidelity. Let $(\mathcal{F}, \iota)$ and $(\mathcal{F}', \iota')$ be in $\mathsf{P}_{\mathsf{G}}(\mathcal{X})$; the claim is that any map $\varphi : \mathcal{F} \to \mathcal{F}'$ automatically intertwines ι and ι'. But this is a question of an equality of two maps $\mathbf{a}^*\mathcal{F} \to \mathbf{a}^*\mathcal{F}'$, both of which give φ under $(\mathbf{e} \times \mathbf{id})^*$ by condition (1) of the definition; so we have equality by Lemma B.2 (2).

Now we show thickness, using the claim about the essential image. Let $\mathcal{F} \in \mathsf{P}(\mathcal{X})$ with $\mathbf{a}^*\mathcal{F} \simeq \mathbf{p}^*\mathcal{F}$, and let $\mathcal{G} \in \mathsf{P}(\mathcal{X})$ be a subquotient of $\mathcal{F}$. By [BBD] Corollary 4.2.6.2, there is a $\mathcal{G}' \in \mathsf{P}(\mathcal{X})$ with $\mathbf{p}^*\mathcal{G}' \simeq \mathbf{a}^*\mathcal{G}$; but applying $\mathbf{e}^*$ gives that $\mathcal{G}' \simeq \mathcal{G}$, showing that $\mathbf{a}^*\mathcal{G} \simeq \mathbf{p}^*\mathcal{G}' \simeq \mathbf{p}^*\mathcal{G}$ and thereby finishing the proof. $\square$

Corollary B.6. *If* G *is connected and* $\mathcal{X}$ *is a* G*-space, then* $\mathsf{P}_{\mathsf{G}}(\mathcal{X})$ *is artinian and noetherian.*

Proof. Clear from the corresponding fact for $\mathsf{P}(\mathcal{X})$ ([BBD] Theorem 4.3.1). $\square$

Because of this proposition, if G is connected, we will often identify $\mathsf{P}_{\mathsf{G}}(\mathcal{X})$ with its essential image in $\mathsf{P}(\mathcal{X})$; so we will speak of an $\mathcal{F} \in \mathsf{P}(\mathcal{X})$ as being G-equivariant, i.e. as satisfying $\mathbf{a}^*\mathcal{F} \simeq \mathbf{p}^*\mathcal{F}$.

Since we are primarily interested in connected groups, one might ask why we use the more complicated original definition, rather than simply defining $\mathsf{P}_{\mathsf{G}}(\mathcal{X})$ to be the appropriate full subcategory of $\mathsf{P}(\mathcal{X})$ as indicated by this proposition. Leaving aside philosophical objections to this alternate approach, the most basic reason is that even if our original group is connected, non-connected groups can arise from it as stabilizers of actions, and must be considered: see Proposition B.13. Now, in what follows, the reader will find many propositions where the group is assumed to be connected, and many where it is not; this dichotomy is never arbitrary, but in fact dictated by pragmatism: if we will require the proposition in the non-connected case, we will deal with the non-connected case; if not, we will make our lives easier and invoke the preceding proposition.

We also would like to caution that the forgetful functor to $\mathsf{P}(\mathcal{X})$ is not necessarily Serre, i.e. (in the above language) an extension of G-equivariant perverse sheaves need not be G-equivariant; we will now pause to give a specific example of this, and talk a bit about extensions of equivariant perverse sheaves.

B.2. **Extensions of equivariant perverse sheaves.** We recall the following lemma, which is our starting-point.

Lemma B.7. *Let* D *be a triangulated category, and* P *the heart of a t-structure on* D*. Then for* $X, Y \in \mathsf{P}$*, we have* $\mathsf{P}^1(X, Y) = \mathsf{D}^1(X, Y)$*, i.e., extensions in* P *of* Y *by* X *are classified by maps* $X \to Y[1]$ *in* D.

Proof. Follows simply from [BBD] Theorem 1.3.6 and Corollary 1.1.10. □

We note that the analogous statement is false for higher extensions, as the following example from algebraic topology shows. Let S be the homotopy category of spectra, which is triangulated via the usual suspension and cofiber sequences. Then S admits a t-structure given by the stable homotopy groups (the verification of the t-structure axioms uses the technique of attaching cells to kill homotopy groups). Its heart is the category of Eilenberg-Maclane spectra, which is equivalent to the category of abelian groups, and the maps $\mathsf{S}^n(K(A,m),K(A,m))$ in question are exactly what give stable cohomological operations. For instance, the previous lemma applied to the current situation gives the Bockstein operations associated to an extension of abelian groups, and the existence of Steenrod operations shows that the lemma does not hold for higher extensions (which are all trivial for abelian groups).

Now we give the desired classification of extensions; it is as simple as one can reasonably wish.

Proposition B.8. *Let G be an algebraic group, and $\mathcal{X}$ a G-space. Assume, for simplicity, that G is connected. Then for $\mathcal{F},\mathcal{G} \in \mathsf{P}_{\mathsf{G}}(\mathcal{X})$, the extensions $\mathsf{P}^1_{\mathsf{G}}(\mathcal{F},\mathcal{G})$ are classified by the maps $\mathcal{G} \to \mathcal{F}[1]$ in $\mathsf{D}(\mathcal{X})$ which are equivariant in the sense that they intertwine $\iota_{\mathcal{G}}$ and $\iota_{\mathcal{F}}[1]$.*

Proof. Lemma B.7 gives the statement when we remove the subscripts indicating G-equivariance; by Proposition B.5, then, we need only see that, for a map $\varphi : \mathcal{F} \to \mathcal{G}[1]$ in $\mathsf{D}(\mathcal{X})$, the corresponding extension of $\mathcal{F}$ by $\mathcal{G}$ is G-equivariant if and only φ itself is G-equivariant, i.e. intertwines $\iota_{\mathcal{F}}$ and $\iota_{\mathcal{G}}[1]$. This is a simple exercise in triangulated categories using the ideas in the proof of Proposition B.5. □

Now we give an example showing that the forgetful functor $\mathsf{P}_{\mathsf{G}}(\mathcal{X}) \to \mathsf{P}(\mathcal{X})$ is not necessarily Serre. By the preceding proposition, we need to give a non-equivariant $\mathcal{F} \to \mathcal{G}[1]$. For this, take $\mathsf{G} = \mathsf{G_m}$, the multiplicative group, and $\mathcal{X} = \mathsf{A}^1$ with the obvious action. If $j : \mathsf{G_m} \to \mathsf{A}^1$ is the inclusion, then $j_*\mathbf{Q}_{\mathsf{G_m}}[1]$ and $\mathbf{Q}_{\mathsf{A}^1}[1]$ are both equivariant and perverse; but any map $\mathbf{Q}_{\mathsf{A}^1}[1] \to j_*\mathbf{Q}_{\mathsf{G_m}}[2]$ classifying a nontrivial cohomology class in $H^1(\mathsf{G_m})$ will be non-equivariant, because such a class pulls back to two different things under the action and projection maps $\mathsf{G_m} \times \mathsf{G_m} \to \mathsf{G_m}$.

B.3. **Change-of-group.** Thus far we have dealt with a fixed group G; now we will vary G and consider relations between the resulting categories P_{G}. We start with the following:

Proposition B.9. *Let G be a linear algebraic group, $\mathcal{X}$ a G-space, and H a closed normal subgroup of G. If there is a geometric quotient $f : \mathcal{X} \to \mathcal{X}/\mathsf{H}$ which is étale-locally H-principal, then $\mathbf{f}^*$ induces an equivalence of categories $\mathsf{P}_{\mathsf{G}/\mathsf{H}}(\mathcal{X}/\mathsf{H}) \simeq \mathsf{P}_{\mathsf{G}}(\mathcal{X})$.*

Proof. The map from the simplicial variety describing the action of G on $\mathcal{X}$ to the simplicial variety describing the action of G/H on $\mathcal{X}/\mathsf{H}$ (bolded pull-back along which gives our functor) is just given at every stage by modding out by the obvious action of H, a smooth map; thus, using Proposition B.3, we can identify $\mathsf{P}_{\mathsf{G}/\mathsf{H}}(\mathcal{X}/\mathsf{H})$ with a descent-type category associated to the appropriate bisimplicial varitety whose first row is the action simplicial variety for G on $\mathcal{X}$ and whose first column is the simplicial variety associated to the map f as in Proposition B.3. However, there is a *splitting* for this bisimplicial variety, induced by the closed immersion $\mathcal{X} \times_{\mathcal{X}/\mathsf{H}} \mathcal{X} \to \mathsf{G} \times \mathcal{X}$ given by $(x,x') \mapsto (h,x')$ if $hx' = x$. (This is not literally a map of varieties, but actually an étale-local map: over trivializing $U \to \mathcal{X}/\mathsf{H}$, it is just the inclusion $\mathsf{H} \times U \to \mathsf{G} \times U$. However, an étale-local map is good enough for us, since we only care about pulling perverse sheaves back along it). We leave it as an exercise to show that the appropriate shifted pullback along this closed immersion is t-exact on the essential image of $\mathbf{a}^*$ and $\mathbf{p}^*$ (using the idea of Lemma B.2 (1)), and to deduce on general terms that the bisimplicial descent category is isomorphic to the simplicial descent category for the first row, as desired. □

This proposition admits a corollary, which uses the notation of [S], page 95—the pertinent point being that, for an algebraic group G, closed subgroup H, and H-space $\mathcal{Y}$, the notation $\mathsf{G}\times^{\mathsf{H}}\mathcal{Y}$ stands for a (geometric) quotient of $\mathsf{G} \times \mathcal{Y}$ by the action of H given by $h \cdot (g, y) = (gh^{-1}, hy)$, and is itself a G-space in the obvious way. This quotient always exists as a variety, at least assuming that G is linear algebraic and $\mathcal{Y}$ is quasi-projective; we sketch the proof, since it's not in [S]. As remarked in [S], Lemma 5.5.8, if the projection $\mathsf{G} \to \mathsf{G}/\mathsf{H}$ has local sections, the space $\mathsf{G} \times^{\mathsf{H}} \mathcal{Y}$ can be constructed by gluing together the $U \times \mathcal{Y}$ where U runs over an open covering of G/H by sets on which $\mathsf{G} \to \mathsf{G}/\mathsf{H}$ has a section. Now, $\mathsf{G} \to \mathsf{G}/\mathsf{H}$ doesn't always have local sections, but it does have étale-local sections, being smooth and surjective; so we get an étale-local construction this way. To make it Zariski-local, note that, by refining these étale maps, we can assume them all to be Galois over their image. This, together with the classical fact that the quotient of a quasi-projective variety by a finite group always exists as a variety, lets us descend to the Zariski case. Now, here is the corollary.

Corollary B.10. *Let G be a linear algebraic group, H a closed subgroup, and $\mathcal{Y}$ a quasi-projective H-space. Then the categories $\mathsf{P}_{\mathsf{G}}(\mathsf{G}\times^{\mathsf{H}}\mathcal{Y})$ and $\mathsf{P}_{\mathsf{H}}(\mathcal{Y})$ are equivalent via mutual pullback to $\mathsf{G} \times \mathcal{Y}$.*

Proof. The map $p : \mathsf{G} \times \mathcal{Y} \to \mathsf{G} \times^{\mathsf{H}} \mathcal{Y}$ is étale-locally H-principal, by the above-outlined construction; so Proposition B.9 implies that $\mathbf{p}^*$ identifies $\mathsf{P}_{\mathsf{G}}(\mathsf{G}\times^{\mathsf{H}}\mathcal{Y})$ with $\mathsf{P}_{\mathsf{G}\times\mathsf{H}}(\mathsf{G} \times \mathcal{Y})$. But the same argument (the roles of G and H being switched) applies to $\mathsf{G} \times \mathcal{Y} \to \mathcal{Y}$, and identifies $\mathsf{P}_{\mathsf{H}}(\mathcal{Y})$ with the same category. □

We finish the change-of-group section with the following result (which illustrates how much easier working with connected groups is; compare with Proposition B.9):

Proposition B.11. *Let G be a connected linear algebraic group, and H a connected normal subgroup. If $\mathcal{X}$ is a G/H-space, then we have $\mathsf{P}_{\mathsf{G}}(\mathcal{X}) = \mathsf{P}_{\mathsf{G}/\mathsf{H}}(\mathcal{X})$ (as full subcategories of $\mathsf{P}(\mathcal{X})$; see Proposition B.5).*

Proof. By [BBD] Proposition 4.2.5, pullback along $\mathsf{G} \times \mathcal{X} \to \mathsf{G}/\mathsf{H} \times \mathcal{X}$ is fully faithful and thick; in particular it reflects isomorphisms. □

B.4. **The case of finitely many orbits.** A case of particular interest for us is when $\mathcal{X}$ has only finitely many G-orbits, since this is satisfied for a reductive G acting via conjugation on its unipotent locus $\mathcal{U}$ (Proposition 2.2 (1)). We start with just one orbit, which can be handled by Corollary B.10.

Lemma B.12. *Let G be a linear algebraic group, $\mathcal{X}$ a homogeneous G-space, and $x_0 \in \mathcal{X}$. Set $\mathsf{H} = \mathrm{Stab}(x_0)$. The category $\mathsf{P}_{\mathsf{G}}(\mathcal{X})$ is equivalent to that of finite-dimensional representations of $\mathsf{H}/\mathsf{H}^\circ$, the group of components of H. Furthermore, each object of $\mathsf{P}_{\mathsf{G}}(\mathcal{X})$ is in fact a G-equivariant local system shifted by $\dim(\mathcal{X})$.*

Proof. Applying Corollary B.10 to the case $\mathcal{Y} = x_0$, a point, we deduce that $\mathsf{P}_{\mathsf{G}}(\mathcal{X})$ is equivalent to $\mathsf{P}_{\mathsf{H}}(\bullet)$ (see [S] Corollary 5.5.4); that this is equivalent to the desired category is immediate straight from the definition. The local system claim follows from the fact that Proposition B.3, Proposition B.9, and Corollary B.10 work just as well for local systems as perverse sheaves, with the same proofs (replacing the bold-faced $\mathbf{f}^*$ with normal f^*); and on a point the notions coincide. □

We remark that the G-equivariant projection $\mathsf{G}/\mathsf{H}^\circ \to \mathsf{G}/\mathsf{H}$, which is Galois with group $\mathsf{H}/\mathsf{H}^\circ$, provides another realization of this equivalence of categories, through the usual correspondence between local systems and representations of the fundamental group.

Now we come to the main proposition of this section. It is an analog of [BBD] Theorem 4.3.1, though slightly more precise.

Proposition B.13. *Let G be a connected linear algebraic group, and $\mathcal{X}$ a G-space having only finitely many orbits. For an orbit $\mathcal{O}$, denote by $\mathrm{Stab}(\mathcal{O})$ the stabilizer of a fixed element of $\mathcal{O}$. Then the simple objects of $\mathsf{P}_{\mathsf{G}}(\mathcal{X})$ correspond exactly to pairs $(\mathcal{O}, \tau)$, where $\mathcal{O}$ is a G-orbit on $\mathcal{X}$ and τ is an irreducible finite-dimensional representation of $\mathrm{Stab}(\mathcal{O})/\mathrm{Stab}(\mathcal{O})^\circ$.*

Proof. For the inclusion $j : \mathcal{Z} \to \mathcal{X}$ of a smooth, connected, locally closed subvariety and a local system $\mathcal{L}$ on $\mathcal{Z}$, denote by $\mathcal{F}_{\mathcal{Z},\mathcal{L}}$ the element $j_{!*}(\mathcal{L}[\dim(\mathcal{Z})])$ of $\mathsf{P}(\mathcal{X})$. Proposition B.5 implies that the forgetful functor $\mathsf{P}_{\mathsf{G}}(\mathcal{X}) \to \mathsf{P}(\mathcal{X})$ preserves and reflects simple objects, so by [BBD] Theorem 4.3.1, the simple objects of $\mathsf{P}_{\mathsf{G}}(\mathcal{X})$ are exactly the $\mathcal{F}_{\mathcal{Z},\mathcal{L}}$ which are G-equivariant, for $\mathcal{L}$ irreducible.

Each pair $(\mathcal{O}, \tau)$ does give rise to such an $\mathcal{F}_{\mathcal{Z},\mathcal{L}}$, namely by taking $\mathcal{Z} = \mathcal{O}$ and $\mathcal{L}$ the local system corresponding to τ as in Lemma B.12. These $\mathcal{F}_{\mathcal{Z},\mathcal{L}} = \mathcal{F}_{\mathcal{O},\tau}$ are indeed G-equivariant, by Proposition B.15; what we need is firstly that they are mutually non-isomoprhic, and secondly that any G-equivariant $\mathcal{F}_{\mathcal{Z},\mathcal{L}}$ is isomorphic to one of them. Both facts will follow from the following lemma, concerning regular perverse sheaves on a variety:

Lemma B.14. *Let $\mathcal{X}$ be a variety, and $\mathcal{Z}$ and $\mathcal{Z}'$ two smooth, connected, locally closed subvarieties of $\mathcal{X}$. If $\mathcal{L}$, respectively $\mathcal{L}'$, is a local system on $\mathcal{Z}$, repsectively $\mathcal{Z}'$, then $\mathcal{F}_{\mathcal{Z},\mathcal{L}} = \mathcal{F}_{\mathcal{Z}',\mathcal{L}'}$ if and only if there is a smooth, connected, locally closed $\mathcal{Z}'' \subseteq \mathcal{Z} \cap \mathcal{Z}'$ dense in both $\mathcal{Z}$ and $\mathcal{Z}'$ with $\mathcal{L}|_{\mathcal{Z}''} = \mathcal{L}'|_{\mathcal{Z}''}$.*

Proof. For "only if", suppose $\mathcal{F}_{\mathcal{Z},\mathcal{L}} = \mathcal{F}_{\mathcal{Z}',\mathcal{L}'}$; so in particular their supports are the same, i.e. $\overline{\mathcal{Z}} = \overline{\mathcal{Z}'}$. Then both $\mathcal{Z}$ and $\mathcal{Z}'$ are open dense in this common closure, and we can take $\mathcal{Z}''$ to be their intersection; we will have the equality of restricted local systems simply because restriction composed with Goresky-Macpherson extension is the identity ([BBD] Proposition 2.1.9).

For "if", we may as well assume $\mathcal{Z}'' = \mathcal{Z}'$ (otherwise we'd apply the same argument twice), and replacing $\mathcal{X}$ with $\overline{\mathcal{Z}}$ we may assume $\mathcal{Z}' \subseteq \mathcal{Z}$ are both open. But then Lemma 4.3.2 of [BBD] shows that the conditions for being the Goresky-Macpherson extension of $\mathcal{L}[\dim(\mathcal{Z})]$ are a subset of those for $\mathcal{L}'[\dim(\mathcal{Z}')]$, so that the sheaves must be equal. □

The "only if" part of the lemma immediately implies that the $\mathcal{F}_{\mathcal{O},\tau}$ are mutually non-isomorphic, since two orbits either agree or are disjoint. On the other hand, let $\mathcal{Z}$ and $\mathcal{L}$ be arbitrary such that $\mathcal{F} := \mathcal{F}_{\mathcal{Z},\mathcal{L}}$ is G-equivariant. Then the support $\overline{\mathcal{Z}}$ of $\mathcal{F}_{\mathcal{Z},\mathcal{L}}$ is G-stable, and hence a finite union of orbits; thus there is an orbit $\mathcal{O}$ open in $\overline{\mathcal{Z}}$. The restriction $\mathcal{F}|_{\mathcal{O}}$ is G-equivariant, perverse, and simple; thus by Lemma B.12 it is the shift by $\dim(\mathcal{O}) = \dim(\mathcal{Z})$

of some local system on $\mathcal{O}$, corresponding to some irreducible τ. Then the lemma gives that $\mathcal{F} = \mathcal{F}_{\mathcal{O},\tau}$, as desired. $\square$

B.5. **Functoriality and the equivariant derived category (Bernstein-Lunts).** We need to discuss functoriality for equivariant perverse sheaves. However, a problem immediately presents itself: just as in the non-equivariant case, the usual functors $f_*, f_!, f^*, f^!$ associated to a map $f : \mathcal{X} \to \mathcal{Y}$ of G-spaces do not necessarily respect perversity; thus, there is no such functoriality on the level of perverse sheaves. This will necessitate the introduction of a larger category, an equivariant constructible bounded derived category $\mathsf{D}_\mathsf{G}(\mathcal{X})$, which is the natural domain for these pushforward and pullback maps. Before turning to this, however, we start with functors which do respect perversity; here the extension to the equivariant case is easy.

Proposition B.15. *Let* G *be an algebraic group. We have the following:*

(1) For a G*-space* $\mathcal{X}$*, a Verdier duality anti-equivalence* $D : \mathsf{P}_\mathsf{G}(\mathcal{X}) \to \mathsf{P}_\mathsf{G}(\mathcal{X})$*;*
(2) For an inclusion $j : \mathcal{Z} \to \mathcal{X}$ *of a locally closed* G*-stable subvariety of a* G*-space, a Goresky-Macpherson exetnsion functor* $j_{!*} : \mathsf{P}_\mathsf{G}(\mathcal{Z}) \to \mathsf{P}_\mathsf{G}(\mathcal{X})$*;*
(3) For two G*-spaces* $\mathcal{X}$ *and* $\mathcal{Y}$*, an external product* $\boxtimes : \mathsf{P}_\mathsf{G}(\mathcal{X}) \times \mathsf{P}_\mathsf{G}(\mathcal{Y}) \to \mathsf{P}_\mathsf{G}(\mathcal{X} \times \mathcal{Y})$.

All of these maps satisfy their usual compatibilities and properties, and the forgetful functor to the non-equivariant situation intertwines these maps with their non-equivariant counterparts.

Proof. In all cases, the construction is immediate from the usual operation commuting with $\mathbf{a}^*$ ands $\mathbf{p}^*$. For (1), this is clear; for (2), see [BBD] page 110 and for (3), [BBD] page 111. One must, of course, also recall that the usual operation does in fact preserve perversity, which in the third case is [BBD] Proposition 4.2.8. The least transparent of the equivariant definitions concerns D, because of its contravariance: we set $D(\mathcal{F}, \iota) = (D\mathcal{F}, D\iota^{-1})$. $\square$

Now we turn to the more complicated $f_*, f_!, f^*, f^!$. As remarked above, we need an equivariant constructible bounded derived category $\mathsf{D}_\mathsf{G}(\mathcal{X})$ in which to work. This is not a simple matter; the naive guesses all have defects. We will not construct it ourselves, but instead refer the reader to [BL], and simply state its main properties, which are all we will need from it.

Here, for simplicity, we assume that G is connected, and think of $\mathsf{P}_\mathsf{G}(\mathcal{X})$ as a full subcategory of $\mathsf{P}(\mathcal{X})$ through Proposition B.5.

Proposition B.16. *Let* G *be a connected linear algebraic group. If* $\mathcal{X}$ *is a* G*-space, there is a triangulated t-category* $\mathsf{D}_\mathsf{G}(\mathcal{X})$ *and a t-exact (forgetful) functor* $\mathbf{F} : \mathsf{D}_\mathsf{G}(\mathcal{X}) \to \mathsf{D}(\mathcal{X})$*; we have the following properties:*

(1) $\mathbf{F}$ *identifies the heart of* $\mathsf{D}_\mathsf{G}(\mathcal{X})$ *with* $\mathsf{P}_\mathsf{G}(\mathcal{X})$*, and moreover reflects perversity;*
(2) Let H *be a closed normal subgroup of* G*. If there is a geometric quotient* $f : \mathcal{X} \to \mathcal{X}/\mathsf{H}$ *which is étale-locally* H*-principal, then there is an equivalence of t-categories* $\mathsf{D}_\mathsf{G}(\mathcal{X}) \simeq \mathsf{D}_{\mathsf{G}/\mathsf{H}}(\mathcal{X}/\mathsf{H})$ *intertwining* $\mathbf{F}$ *and* $\mathbf{f}^*$*;*
(3) If $f : \mathcal{X} \to \mathcal{Y}$ *is a map of* G*-spaces, we have associated functors* $f_*, f_!, f^*, f^!$ *satisfying their usual properties (the pertinent part for us being the adjunctions). Both* $\mathbf{F}$ *and the isomorphism of (2) intertwine these operations with their usual counterparts.*

From these properties we can draw several simple consequences. For instance:

Proposition B.17. *Let* G *be a connected linear algebraic group, and* H *a closed connected subgroup of* G*. If* $\mathcal{X}$ *is a* G*-space, there is a natural restriction functor* $R = R^\mathsf{G}_\mathsf{P} : \mathsf{D}_\mathsf{G}(\mathcal{X}) \to$

$\mathsf{D}_{\mathsf{H}}(\mathcal{X})$, *which* $\mathbf{F}$ *intertwines with the identity* $\mathsf{D}(\mathcal{X}) \to \mathsf{D}(\mathcal{X})$*; moreover,* R *has a right adjoint* $I = I_{\mathsf{H}}^{\mathsf{G}}$.

Proof. The same proof as in Corollary B.10 shows that property (2) above implies $\mathsf{D}_{\mathsf{H}}(\mathcal{X}) \simeq \mathsf{D}_{\mathsf{G}}(\mathsf{G} \times^{\mathsf{H}} \mathcal{X})$; through this isomorphism, we can define $R = \mathbf{a}^*$ and $I = a_*[-\dim(\mathsf{G}/\mathsf{H})]$, where $a : \mathsf{G} \times_{\mathsf{H}} \mathcal{X} \to \mathcal{X}$ is given by the action map (isomorphic to the projection $\mathsf{G}/\mathsf{H} \times \mathcal{X} \to \mathcal{X}$). $\square$

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