Classification and Structure of Periodic Fatou Components

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Abstract

For a given rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, the Julia set consists of those points in $\widehat{\mathbb{C}}$ around which the dynamics of the map is chaotic (a notion that can be defined rigorously), while the Fatou set is defined as the complement. The Fatou set, where the dynamics is well-behaved, is an open set, and one can classify its periodic connected components into five well-understood categories. This classification theorem is the focus of the paper, and we attempt to present its proof in an efficient, self-contained, and well-motivated manner. The proof makes heavy use of methods of hyperbolic geometry on certain open subsets of $\widehat{\mathbb{C}}$. We develop the theory needed to carry out this analysis. One fundamental result that is often used is the Uniformization Theorem, whose proof in the general case would take us far afield from the usual subject matter of complex dynamics. We present a simpler proof for the case of plane domains, which is all that is needed for the Fatou component classification theorem. Finally we show that each of the five types of Fatou components actually occurs, and we present some of the theory associated with the structure of each.

Contents

Chapter 1

Introduction

Dynamics is the study of how systems evolve over time. To study a dynamical system mathematically, we need a rule for determining how the system transforms from one state to another. Then we can investigate how various initial configurations change as the transformation rule is repeatedly applied. The solar system was one of the first dynamical systems to be studied in earnest. Newton's laws of motion and gravitation provide rules for how the planets and other celestial bodies move around. The transformation rule in this case takes the form of a system of differential equations. Dynamical systems also come up in biology, meteorology, other areas of physics, economics, and in many areas of mathematics, such as number theory. In this paper we study holomorphic dynamics of one complex variable, which is concerned with iterating complex analytic functions. These dynamical systems are *discrete* in the sense that the transformation rule is given by a function (rather than, say, a differential equation), and one time interval corresponds to one iteration of the function. One advantage of studying holomorphic dynamics over other forms of dynamics is that complex analysis gives us many powerful tools that would not be available in other contexts.

Interesting results in dynamics are often qualitative. Part of the reason for this is that systems are often sufficiently complicated that precise quantitative results are nearly impossible to come by. One important qualitative notion is *chaos*, which is often vaguely described as "sensitive dependence on initial conditions," meaning that small changes in the initial configuration can result in dramatic changes in the long-term evolution of the system. In holomorphic dynamics we can make this notion into a precise definition through the Fatou and Julia sets. For a given map, the Julia set corresponds to points around which the long-term behavior is chaotic, while the complementary Fatou set corresponds to points around which the long-term behavior is well-behaved. The main object of study will be the Fatou set, but of course understanding the Julia set is often also helpful. The central result described in this paper is a theorem giving a precise classification of the periodic connected components of the Fatou set of a rational map. The key tool that we use to prove the classification theorem is hyperbolic geometry.

1.1 Outline

In chapter 2 we state and prove the Five Possibilities Theorem, which gives a list of the types of periodic Fatou components that a rational map might have. Before proving this, we introduce the hyperbolic metric and discuss various properties of hyperbolic surfaces. We prove the Uniformization Theorem for plane domains, using some ideas from hyperbolic geometry. This result allows us to use hyperbolic metrics on a wide class of spaces. Next, we introduce the Fatou and Julia sets. We then combine the tools and theory that we have developed to prove the Five Possibilities Theorem. This chapter is almost entirely self-contained, requiring only standard material from complex analysis and a touch of differential geometry.

In chapter 3 we show that all five types of Fatou components actually do occur. We investigate the often rich structure of the different types of Fatou components. For Herman rings and Siegel disks (two of the more exotic types of Fatou components), including all details would take us too far afield, so references are given to more complete proofs and to more general results.

1.2 Historical Note

The study of holomorphic dynamics began with the work of Gaston Julia and Pierre Fatou, in the early 20th century. See, for instance, [Julia, 1918] and [Fatou, 1919]. Early practitioners made clever use of Paul Montel's results on normal families [Montel, 1912], which are closely connected with hyperbolic geometry. After an initial flurry of activity, interest faded until the early 1980's when various researchers used computers to make beautiful pictures of sets that arise in holomorphic dynamics, notably the Mandelbrot set, which is closely related to Julia sets. Motivated in part by a desire to understand these often breathtakingly rich images, researchers have quickly amassed a wealth of new results. Many new exciting avenues for future exploration have been opened, and connections to other areas of mathematics have been revealed.

The proof given in this paper of the Five Possibilities Theorem uses only tools that would have been available to Fatou and Julia. Showing that all the five possibilites actually occur requires more sophisticated techniques. It was not until 1979 that Michael Herman [Herman, 1979] was able to construct a rational map with a rotation domain conformally equivalent to an annulus (such a domain is now called a Herman ring). Dennis Sullivan's landmark proof that any Fatou component is eventually periodic (a result originally conjectured by Fatou) uses the technique of quasiconformal deformation [Sullivan, 1985].

1.3 Notation

- $\widehat{\mathbb{C}}$ is the Riemann sphere, i.e. the complex plane $\mathbb{C}$ extended by the point ∞
- $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ is the open disk of radius r
- $\mathbb{D} = \mathbb{D}_1 = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk (sometimes just referred to as the disk)
- $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle
- $f^{\circ k} = \underbrace{f \circ f \circ \cdots \circ f}_{k}$ is the kth iterate of f, where $f : S \to S$ is a function from some set S to itself. We will adopt the convention that if $k = 0$, then $f^{\circ k}$ is the identity function on S.
- $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ is the punctured complex plane
- $\mathbb{D}^* = \mathbb{D}\backslash\{0\}$ is the punctured unit disk
- $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ is the upper half-plane (or just half-plane)

1.4 Acknowledgements

I would like to thank my adviser Sarah Koch, for guiding me through the reading and writing processes, and diligently answering my many questions. I also wish to thank

Curt McMullen, for advice on choosing a suitable senior thesis topic.

This thesis also owes a great deal to the beautiful expository work of John Milnor [Milnor, 2006], which has, at least for me, made the subject of complex dynamics approachable.

Chapter 2

Five Possibilities

2.1 Hyperbolic geometry

We will make frequent use of *geometry* to understand the complex structures on the spaces that we study. We would like to define geometric notions in way that is invariant under the action of complex analytic automorphisms, and we will find that for many of the spaces that arise, this is possible.

2.1.1 Riemannian and conformal metrics

To specify a geometry on a smooth manifold M, we must come up with some notion of distance, which will then allow us to make sense of concepts such as straight lines (geodesics), contracting maps, etc. We accomplish this by specifying a *Riemannian metric*, which consists of a (positive definite) inner product on each tangent space, such that the inner products vary smoothly (for the precise meaning of smoothness in this context, see [Lee, 2003]). Thus a Riemannian metric will allow us to assign lengths to tangent vectors. Concretely, in the case where the manifold has two real dimensions (the class of manifolds that we are interested in), we can write the metric at a point z as an expression of the form

$$ds^2 = a_{11}dx^2 + 2a_{12}dxdy + a_{22}dy^2,$$

where $[a_{ij}]$ is a positive-definite matrix that depends smoothly on the point z. We will restrict our study to *conformal metrics*, those that are invariant under local rotations, i.e. the length of a tangent vector at a point is unchanged by a rotation (in a small neighborhood) about that point. This condition implies that $a_{12} = 0$, and $a_{11} = a_{22}$, and so we can write $ds = \gamma(z)|dz|$, where $\gamma(z)$ is a smoothly varying function of z, that is everywhere positive.

In this paper, we will be interested in manifolds S that are Riemann surfaces (i.e. complex manifolds of one complex dimension). Given a holomorphic map $g : S_1 \to S_2$ of Riemann surfaces, if we have a conformal metric $ds = \gamma(z)|dz|$ on S_2, we can define a metric $f^*(ds)$ on S_1, the *pullback* of γ along f, by

$$f^*(ds) = \gamma(g(z))|dg(z)| = \gamma(g(z)) \cdot |g'(z)||dz|.$$

We will study metrics that do not change when pulled back along conformal automorphisms.

Definition 2.1.1. A conformal metric $ds = \gamma(z)|dz|$ on $S \subset \widehat{\mathbb{C}}$ is said to be *conformally invariant* if for every $f \in \mathrm{Aut}(S)$,

$$\gamma(z)|dz| = \gamma(f(z)) \cdot |f'(z)| \cdot |dz|.$$

2.1.2 The disk and half-plane models

Our next goal is to study conformally invariant metrics on the unit disk $\mathbb{D} \subset \widehat{\mathbb{C}}$. We first recall some of the relevant complex analysis.

Theorem 2.1.2 (Riemann Mapping Theorem)**.** *Any proper open subset of $\mathbb{C}$ that is simply connected is conformally isomorphic to the unit disk $\mathbb{D}$.*

Proof. See [Ahlfors, 1979]. ∎

Since we are interested in conformally invariant metrics, the relevant behavior of any set that satisfies the conditions of the above theorem can be understood by studying the behavior on $\mathbb{D}$. Nevertheless it is sometimes useful to consider other models, in particular the half-plane $\mathbb{H}$. The conformal isomorphism $\mathbb{D} \to \mathbb{H}$ is given by the Möbius transformation

$$z \mapsto i\frac{1+z}{1-z} \tag{2.1}$$

Since this map is particularly simple, it is often useful to move back and forth between $\mathbb{D}$ and $\mathbb{H}$ when studying conformally invariant metrics. For instance, the natural expressions for the conformal automorphism groups of the two spaces (which of course are ultimately isomorphic groups) look somewhat different, and one expression or the other may be easier to use in a particular situation.

The disk is often handy to work with because of the following basic tool which comes up over and over again:

Lemma 2.1.3 (Schwarz). *Let $f : \mathbb{D} \to \mathbb{D}$, with $f(0) = 0$. Then $|f(z)| \leq |z|$ for all z and $|f'(0)| \leq 1$. If equality holds in any of these expressions, then f is a conformal automorphism given by $f(z) = e^{i\theta}z$ for some $\theta \in [0, 2\pi)$.*

Proof. Consider the function

$$g(z) = \begin{cases} \frac{f(z)}{z} & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0, \end{cases}$$

which is holomorphic on $\mathbb{D}$. Note that $\limsup_{|z|\to 1} |g(z)| \leq 1$, so by the maximum modulus principle, g is bounded in magnitude by 1. If $|f(z)| = |z|$ for some z or $f'(0) = 1$, then, again by the maximum modulus principle, g must be constant, hence $f(z) = e^{i\theta}z$ for some θ. Otherwise, $|f(z)| \leq |z|$ for all z, and $|f'(0)| \leq 1$. ■

We now use this result to determine the conformal automorphism group of the disk.

Proposition 2.1.4. *A map f belongs to* $\mathrm{Aut}(\mathbb{D})$ *iff it can be written as a Blaschke factor, i.e. a map of the form*

$$B_{a,\theta}(z) = e^{i\theta}\frac{z - a}{\bar{a}z - 1}$$

for some $a \in \mathbb{D}$, and $\theta \in [0, 2\pi)$. We will adopt the notation $B_a = B_{a,0}$.

Proof. First we show that Blaschke factors are in fact automorphisms. Note that each Blaschke factor $B_{a,\theta}$ extends to a Möbius transformation $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ in the obvious way. Hence $B_{a,\theta}$ is clearly injective. When $|z| = 1$, we have $1/z = \bar{z}$ and

$$|F(z)| = \frac{|z - a|}{|z(\bar{a} - 1/z)|} = \frac{|z - a|}{|\bar{a} - \bar{z}|} = 1.$$

Thus $F(\partial\mathbb{D}) \subset \partial\mathbb{D}$. Similarly, we find that $F^{-1}(\partial\mathbb{D}) \subset \partial\mathbb{D}$, since F^{-1} can also be written as the extension to $\widehat{\mathbb{C}}$ of a Blaschke factor. We also note that $F(0) = e^{2\pi i\theta}a \in$

$\mathbb{D}$. These facts together imply that F maps the unit disk into the unit disk, as does F^{-1}. Hence $B_{a,\theta}$ is a conformal automorphism.

For the converse, suppose that we are given a conformal automorphism f. There must be some $a \in \mathbb{D}$ with $f(a) = 0$. Then $g := f \circ B_a$ is an automorphism with $g(0) = 0$. Let $h = g^{-1}$. Then $g'(0) \cdot h'(0) = 1$, so one of $g'(0), h'(0)$ has magnitude at least 1. Applying the Schwarz lemma, we see that $|g'(0)| = |h'(0)| = 1$, and hence $g(z) = e^{i\theta} z$ for some θ. So $f = g \circ B_a^{-1} = e^{i\theta} B_a^{-1}$, and f is a Blaschke factor, as desired. ∎

Remark 2.1.5. Any element $f \in \mathrm{Aut}(\mathbb{D})$ extends to a Möbius transformation $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ that commutes with the inversion map $z \mapsto 1/\overline{z}$ (this follows from the fact that F fixes the boundary $\partial\mathbb{D}$, and inversion is the identity on $\partial\mathbb{D}$). It then follows that the set of fixed points of F in $\overline{\mathbb{D}}$ consists of either

(i) a single point in $\mathbb{D}$ (in which case f is called *elliptic*)

(ii) a single point in $\partial\mathbb{D}$ (*parabolic*)

(iii) exactly two points in $\partial\mathbb{D}$ (*hyperbolic*).

A similar discussion applies to elements of $\mathrm{Aut}(\mathbb{H})$.

Proposition 2.1.6. *A map f belongs to* $\mathrm{Aut}(\mathbb{H})$ *iff it can be written as* $f(z) = \frac{az+b}{cz+d}$ *with $ad - bc = 1$ and $a, b, c, d \in \mathbb{R}$. It follows that* $\mathrm{Aut}(\mathbb{H}) \cong \mathrm{PSL}_2(\mathbb{R})$.

Proof. Note that maps of the form $f(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$, $ad - bc = 1$ are exactly those Möbius transformations which preserve the real axis. The rest of the proof is similar to the proof of the previous theorem. See [Milnor, 2006]. ∎

We now return to the subject of conformally invariant metrics. It is easiest in this case to work with $\mathbb{H}$. Suppose that $\gamma(z)|dz|$ is a conformally invariant metric on $\mathbb{H}$. In particular, pullbacks along automorphisms of the form $g(z) = az + b$ with $a, b \in \mathbb{R}$, $a > 0$ must preserve the metric. Along g, the metric pulls back to $\gamma(g(z)) \cdot |g'(z)| \cdot |dz|$. Thus we must have $\gamma(az + b) \cdot a|dz| = \gamma(z)|dz|$ for all z. Setting $z = i$, gives $\gamma(ai + b) = \gamma(i)/a$. We can assume that $\gamma(i) = 1$ (since we will regard two metrics that differ by a positive constant scale factor as equivalent). Hence we must have, for all z,

$$\gamma(z)|dz| = \frac{|dz|}{\mathrm{Im}(z)}.$$

Now we need to verify that this metric is in fact invariant under all automorphisms of $\mathbb{H}$. Consider an arbitrary $f \in \mathrm{Aut}(\mathbb{H})$. We will show at for given point $w \in \mathbb{H}$, the metric and its pullback along f agree at the point w. Since the affine automorphisms of the form $z \mapsto az+b$ considered above form a transitive subgroup of $\mathbb{H}$, we can write $f = g \circ h$, where g is affine and h fixes w. By the above, the pullback along g preserves the value of the metric at w. By transforming to the disk, applying Lemma 2.1.3 (Schwarz), and then moving back to $\mathbb{H}$, we see that $|h'(w)| = 1$. Together with the assumption that w is a fixed point of h, this implies that pullback along h also preserves the metric at w, and it follows that pullback along $f = g \circ h$ does as well. Since this holds for all w, we conclude that pullback along f preserves the value of the metric at all points, as desired. We now summarize these results:

Proposition 2.1.7. *There is a unique (up to multiplication by a positive constant) conformally invariant metric on $\mathbb{H}$, given by*

$$ds = \frac{|dz|}{\mathrm{Im}(z)}.$$

Using the map (2.1), it is easy to translate this result to $\mathbb{D}$:

Proposition 2.1.8. *There is a unique (up to multiplication by a positive constant) conformally invariant metric on $\mathbb{D}$, given by*

$$ds = \frac{|dz|}{1 - |z|^2}.$$

These metrics on simply connected domains are called the *Poincaré metrics.* They are powerful tools. For the rest of the paper when we talk about the geometry on a surface, it will be assumed that the Poincaré metric is being used, unless explicitly stated otherwise. In section 2.2 we will see that many open domains $U \subset \widehat{\mathbb{C}}$ admit covering maps from $\mathbb{D}$, and that the metric on $\mathbb{D}$ descends to a conformally invariant metric on U.

2.1.3 Hyperbolic distance and geodesics

Any metric ds on U naturally gives rise to a distance function $d_U : M \times M \to \mathbb{R}_{\geq 0}$ defined by

$$d_U(x, y) = \inf_{\gamma} \int_{\gamma} ds,$$

where the infimum is taken over all piecewise-smooth paths $\gamma : [0,1] \to M$ with $\gamma(0) = x$ and $\gamma(1) = y$. We define the *hyperbolic length* of the path γ to be the quantity $\int_\gamma ds$. It is easy to verify that the above notion of distance is in fact a metric in the sense of metric spaces, and that the topology induced by the distance function agrees with the standard topology on U (this is true in general for Riemannian metrics on smooth manifolds - see [Lee, 2003] for details).

A map which preserves this notion of distance is said to be an *isometry* for the metric.

A *geodesic segment* in U is a path that is locally the shortest path between two points. That is, a geodesic is a piecewise smooth map γ from an interval $I \subset \mathbb{R}$ to U, such that given any $t \in I$, there is a small compact interval $[a,b] \subset I$ containing t such that $d_U(\gamma(a), \gamma(b)) = \int_{\gamma|_{[a,b]}} ds$. We will call a geodesic segment a *geodesic line* if it is maximal, i.e. there is no other geodesic of which it is a proper subset.

We now specialize to the case $U = \mathbb{H}$.

Proposition 2.1.9. *The geodesic lines of $\mathbb{H}$ in its Poincaré metric consist of circles/lines that are orthogonal to $\partial\mathbb{H}$. That is, they are circular arcs that intersect the real axis orthogonally at two points, as well as lines orthogonal to the real axis.*

Proof. We need to find the geodesic segments with respect to the metric $ds = |dz|/\operatorname{Im}(z)$. Given two points w_1, w_2, we can find an $f \in \operatorname{Aut}(\mathbb{H})$ taking w_1 to i and w_2 to ki for some $k \geq 1$. This is accomplished by first moving w_1 to i, and then performing a "rotation" about w_1 (i.e. apply a map that would be a rotation if we conjugated to the unit disk). Given a piecewise smooth path γ from i to ki, we have

$$\int_\gamma ds = \int_\gamma \frac{|dz|}{\operatorname{Im}(z)} \geq \int_i^{ki} \frac{dy}{y},$$

with equality iff γ runs along the imaginary axis. Hence the unique geodesic line containing i and ki is the set $i\mathbb{R}_+$, which is orthogonal to $\partial\mathbb{H}$. Since f is a hyperbolic isometry, the preimage of this geodesic line is a geodesic line containing w_1 and w_2. Since f, f^{-1} extend to Möbius transformations, they take circles/lines to circles/lines. The maps are also conformal, so they preserve orthogonality. Thus the geodesic line connecting w_1, w_2 is an arc of a circle/line intersecting $\partial\mathbb{H}$ orthogonally. ■

This result would have been of great interest to early geometers. The upper half-plane with hyperbolic geodesic lines gives a model of geometry that satisfies the first four of Euclid's postulates. Yet the model does not satisfy the fifth ("parallel") postulate; given a geodesic line L and a point p not on it, there are in general infinitely many geodesic lines through p not intersecting L. Thus the model provides a proof that the fifth postulate cannot be derived from the others.

As usual, it is easy to translate these results to the disk:

Proposition 2.1.10. *The geodesic lines of $\mathbb{D}$ in its Poincaré metric consist of circles/lines that are orthogonal to $\partial\mathbb{D}$. That is, they are circular arcs that intersect $\partial\mathbb{D}$ orthogonally at two points, as well as diameters of $\partial\mathbb{D}$. See Figure 2.1.*

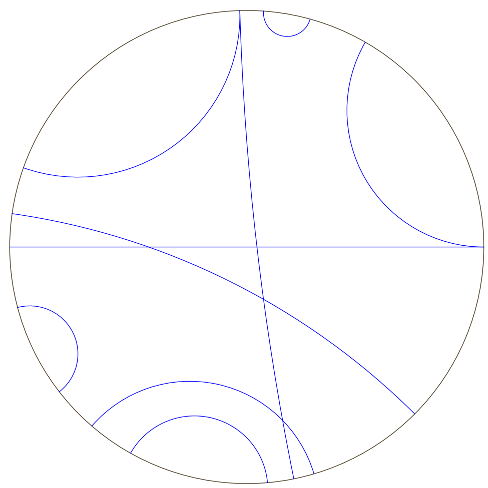

Figure 2.1: Assorted geodesics in $\mathbb{D}$.

2.2 Uniformization

The goal of this section is the following central result which will allow us to adapt the hyperbolic geometry that we developed in the previous section to a wide variety of spaces.

Theorem 2.2.1 (Uniformization of plane domains)**.** *Let U be a connected open subset of $\widehat{\mathbb{C}}$. Then the universal covering space of U is conformally isomorphic to either $\widehat{\mathbb{C}}, \mathbb{C}$, or $\mathbb{D}$.*

In fact the above is still true if we replace U by any connected Riemann surface, but the proof in this higher level of generality is much more difficult - see [Ahlfors, 1973].

As a consequence of the theorem, it will follow that "most" subsets of $\mathbb{C}$ have universal covers isomorphic to the disk.

2.2.1 Covering the triply punctured sphere

There are no subsets of $\mathbb{C}$ whose universal cover is isomorphic to $\widehat{\mathbb{C}}$, and the only subsets of $\mathbb{C}$ covered by $\mathbb{C}$ are $\mathbb{C}$ and $\mathbb{C}^*$. This is proved by noting that any space covered by $\widehat{\mathbb{C}}$ or $\mathbb{C}$ can be obtained by taking the quotient of one of these spaces by a group of analytic automorphisms acting freely and properly discontinuously on that space. There are no such non-trivial subgroups of the automorphism group of $\widehat{\mathbb{C}}$, since every analytic automorphism is a Möbius transformation, which has a fixed point, and thus no non-trivial group of automorphisms can act freely. For $\mathbb{C}$ the non-trivial subgroups acting properly discontinuously are isomorphic to $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$, which give covers of $\mathbb{C}^*$, and tori, respectively (but tori cannot be embedded in $\mathbb{C}$).

Thus to prove Theorem 2.2.1, it suffices to prove the theorem in the case $U \subset \mathbb{C}\backslash\{0,1\}$. As a warm-up to the proof of the full uniformization theorem, we geometrically construct the covering map from $\mathbb{D}$ to $\mathbb{C}\backslash\{0,1\}$.

Theorem 2.2.2. *There exists a covering map $p : \mathbb{D} \to \mathbb{C}\backslash\{0,1\}$.*

Proof. Our proof makes use of the hyperbolic geometry developed in the previous section. The strategy is to first show that one can tile the disk with hyperbolic triangles, and then use the union of two adjacent triangles as the fundamental domain of p. There are alternative constructions using the modular function (see Theorems 7 and 8 of [Ahlfors, 1979]).

Constructing the tiling: Consider a triangle T in $\mathbb{D}$ (with respect to the hyperbolic metric, i.e. the sides are hyperbolic geodesics) with vertices at $-1, 1, i$. Let S_1, S_2, S_3 be the sides of T, and R_1, R_2, R_3 be hyperbolic reflections through these sides. For instance, the reflection through the side connecting -1 and 1 is just given by the map $z \mapsto \bar{z}$. We will tile the whole disk with triangles by repeatedly reflecting T across its sides. Figure 2.2 shows the first step in this process, while Figure 2.3 shows the result after many iterations.

We now show that this process does in fact result in a tiling of the whole disk. Let Γ be the group of reflections through sides of triangles in the tiling. This group is generated by R_1, R_2, R_3. Now suppose we are given a point z that we wish to show is contained in some triangle of the tiling. Let p be an arbitrary point in the interior of T.

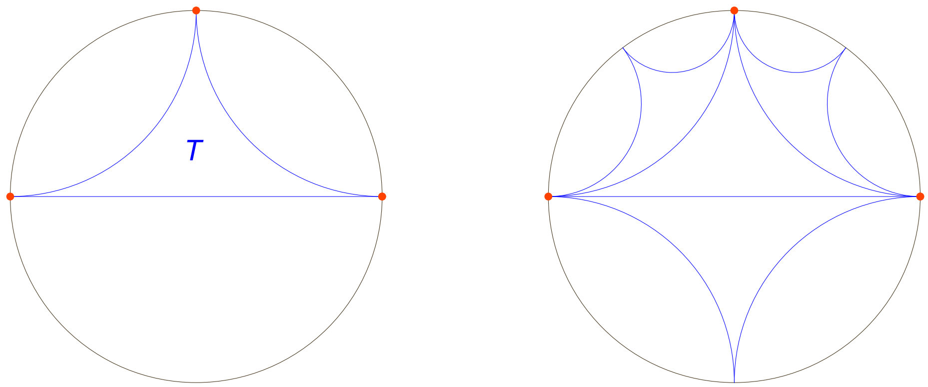

Figure 2.2: An ideal triangle T, and its reflections across its sides.

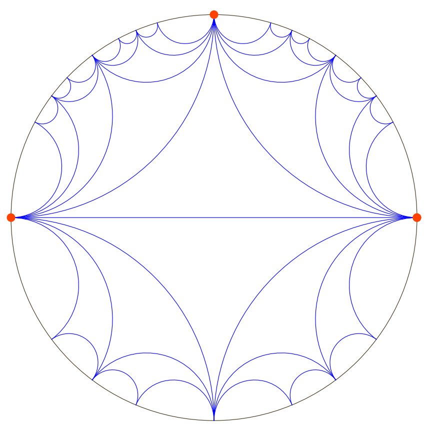

Figure 2.3: Hyperbolic tiling of the disk.

Draw the geodesic segment L from p to z. Now imagine another path L' contained entirely within T, defined as follows. Start at p and follow L until we get to the boundary of T. Then reflect across the boundary and continue on a geodesic segment until we hit another side of T, in which case we reflect again, etc. Let $S_{i(1)}, S_{i(2)}, \ldots$ be the sequence of sides reflected across. This sequence will be finite, since the Poincaré distance $d_{\mathbb{D}}(p, z)$ is finite, and the length of the path L will equal the length of the path L' since each reflection is a hyperbolic isometry. See Figure 2.4 for an illustration of this process. Let n be the length of the sequence of sides reflected across, and let z' be the endpoint of L'. It is then clear that $R_{i(1)} \circ R_{i(2)} \cdots \circ R_{i(n)}(z') = z$, and hence $z \in \Gamma(T)$.

Completing the proof: Let $\Gamma_0 \subset \Gamma$ be the subgroup consisting of compositions

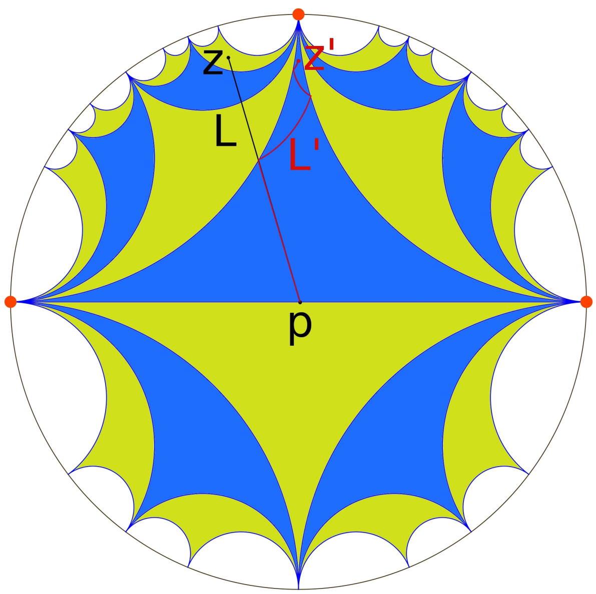

Figure 2.4: The paths L and L'.

of two reflections. This is the *orientation preserving* subgroup, and its elements, in addition to being hyperbolic isometries, are also analytic maps $\mathbb{D} \to \mathbb{D}$ (this follows from the fact that the conjugating map $z \mapsto \bar{z}$ commutes with polynomials and hence analytic functions).

Consider the map $f : \mathbb{D} \to U$ defined as the quotient map obtained from $\mathbb{D}$ by modding out by the action of Γ_0. It is clear that f is a analytic covering map, since Γ_0 is group of analytic automorphisms acting freely and properly discontinuously. A fundamental domain for the action of Γ_0 is $F = T \cup (-T)$, where $-T$ is the triangle obtained by reflecting T through its side connecting -1 and 1 (we exclude the points $-1, 1, i$ from F). In fact, recall that $\Gamma(T) = \mathbb{D}$, hence $\Gamma_0(F) = \mathbb{D}$. No two points in the interior of F are in the same orbit under the action of T. We conclude that F is a fundamental domain for Γ_0.

Note that the side $S_{-1,i}$ of T connecting -1 to i is identified with its image under the reflection taking T to $-T$, and similarly for the side $S_{1,i}$. It follows that, topologically, U is a triply punctured sphere. See Figure 2.5.

We would be done if we could prove that any complex structure on the topological triply punctured sphere is conformally equivalent to the standard complex structure on $\mathbb{C}\backslash\{0, 1\}$ induced from $\widehat{\mathbb{C}}$. Instead, motivated by the above discussion, we explicitly construct a covering map $p : \mathbb{D} \to \mathbb{C}\backslash\{0, 1\}$. We start by considering the Riemann

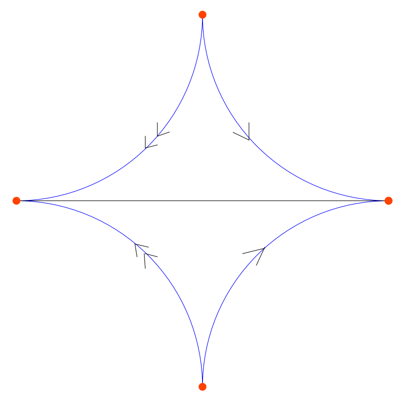

Figure 2.5: Identification of the edges of the fundamental domain.

mapping $p_0 : T \to \mathbb{H}$ which can be chosen such that it extends to a map from the closure of T to the closure of $\mathbb{H}$ (as subsets of the Riemann sphere), taking $-1 \mapsto 0$, $1 \mapsto 1$ and $i \mapsto \infty$. We can get an explicit form for this map, using the generalization of the Schwarz-Christoffel formula to spherical polygons (though the formula will be in terms of integrals that may be difficult to evaluate - see [Nehari, 1952]). We can then use the Schwarz reflection principle to extend the domain of p_0 to $-T$, the reflection of T across the x-axis. Similarly, we can use the reflection principle through the other two sides of T. We continue in this manner, extending p_0 to a map p on the whole disk. There are pairs of different sequences of reflections whose composition is the same, but the choice of sequence we use to extend the domain of p will not affect the result. The image of the extended map is $\mathbb{C}\backslash\{0, 1\}$, and it is manifestly a covering map. ■

2.2.2 Proof of uniformization of plane domains

The discussion in this section is similar to the presentation in [Fisher et al., 1988].

Proof of Theorem 2.2.1 (Uniformization of plane domains). As remarked above, we can assume that $U \subset \mathbb{C}\backslash\{0, 1\}$. We will use the covering map $p : \mathbb{D} \to \mathbb{C}\backslash\{0, 1\}$ constructed above to reduce to the case where $U \subset \mathbb{D}$. Consider a connected component V of $p^{-1}(U) \subset \mathbb{D}$. Note that p restricts to a covering map $V \to U$. Hence U and $V \subset \mathbb{D}$ must have isomorphic universal covers.

It is also clear that we can assume $0 \in U$. Thus we will be done if we can prove the following proposition.

Proposition 2.2.3. *Suppose that $U \subset \mathbb{D}$ and that $0 \in U$. Then the universal covering space of U is isomorphic to $\mathbb{D}$.*

We first give a sketch of the proof. We construct a sequence of spaces and covering maps $U = U_0 \xleftarrow{p_1} U_1 \xleftarrow{p_2} U_2 \xleftarrow{p_3} \cdots$ where the $U_i \subset \mathbb{D}$ are larger and larger domains, each containing 0, that approach the whole disk. This is achieved by taking the maps to be suitable modifications of the function $\mathrm{sq}(z) = z^2$ (avoiding the branch point at 0 so as to get a bona fide covering map). Since $\mathrm{sq}(V)$ "contracts" V in some sense, we get the desired expansion properties of the sequence $U_0, U_1, \ldots$. Some subsequence of the sequence of compositions $p_1, p_1 \circ p_2, p_1 \circ p_2 \circ p_3$ will converge to a holomorphic map $p : \mathbb{D} \to U$. We won't actually be able to show that this is a covering map, but we can show that the abstract universal covering map $\tilde{U} \to \mathbb{D}$ factors through p in such a way as to induce an isomorphism $\tilde{U} \cong \mathbb{D}$, which will complete the proof.

We now fill in the details. For a domain $V \subset \mathbb{D}$ containing 0, we define the *inner radius* $\mathrm{in}(V)$ to be the radius of the largest disk with center at the origin whose interior is contained entirely within V, i.e.

$$\mathrm{in}(V) := \sup\{r : \mathbb{D}_r \subset V\},$$

This is the notion of "size" of a domain that we will use.

Suppose we have constructed U_{n-1}. We construct U_n as follows. Let $a_n \neq 0$ be a point in ∂U_{n-1} that is as close to the origin as any other point in ∂U_{n-1}. Note that $|a_n| = \mathrm{in}(U_{n-1})$. We will define a map $\hat{p}_n : (\mathbb{D}, 0) \to (\mathbb{D}, 0)$ (i.e. taking 0 to 0) that is a modification of the square root function, chosen to be branched over a_n instead of 0. Recall that the Blaschke factor map B_a of Proposition 2.1.4 exchanges the points $a, 0$. Hence we define

$$\hat{p}_n = B_{a_n} \circ \mathrm{sq} \circ B_{\sqrt{a_n}},$$

where the square root $\sqrt{a_n}$ can be chosen arbitrarily. Now consider the open set $\hat{p}_n^{-1}(U_{n-1})$. This set has either one or two connected components. Define U_n to be the connected component containing 0, and let $p_n : U_n \to U_{n-1}$ be the restriction of $\hat{p}_n$ to U_n. Note that U_n is a covering map of degree 1 or 2.

Proposition 2.2.4. *For each $n \geq 1$,*

$$\mathrm{in}(U_n) \geq h(\mathrm{in}(U_{n-1})),$$

where $h : (0,1) \to [0,1]$ is a continuous function satisfying $h(r) > r$ for all r.

Proof. Note that U_{n-1} contains an open disk of radius $r = |a_n|$, and U_n contains the component of $\hat{p}_n^{-1}(\mathbb{D}_r)$ containing the origin. Hence U_n contains an open disk of radius $h(r) := \inf\{|z| : \hat{p}_n(z) = r\}$. This function is continuous. Applying Lemma 2.1.3 (Schwarz) to $\hat{p}_n$, we see that $|\hat{p}_n(z)| \leq |z|$ for all z, and in fact, since $\hat{p}_n$ is not an isomorphism,we have the strict inequality $|\hat{p}_n(z)| < |z|$ for $z \neq 0$. Hence for each z with $\hat{p}_n(z) = r$, we have $|z| > r$. Since $h(r)$ is defined as the infimum of these values of $|z|$ on a compact set, it follows that $h(r) > r$, and

$$\mathrm{in}(U_n) \geq h(r) = h(a_n) = h(\mathrm{in}(U_{n-1})).$$

■

Proposition 2.2.5. *We have $\mathrm{in}(U_n) \to 1$ as $n \to \infty$.*

Proof. Since $\mathrm{in}(U_n)$ increases with n and is bounded above by 1, the value $A := \lim_{n\to\infty} \mathrm{in}(U_n)$ is defined. Suppose for the sake of contradiction that $A < 1$. It follows that $\mathrm{in}(U_n) \leq A$ for all n. Using the function h given by the previous proposition, we have $h(A) = A + 2\epsilon$, for some $\epsilon > 0$. By continuity of h, for some $\delta > 0$, we have $h(x) > A + \epsilon$ for all $x \in [A - \delta, A + \delta]$. For sufficiently large n, $\mathrm{in}(U_n) > A - \delta$, hence $\mathrm{in}(U_n) \in [A - \delta, A + \delta]$. But then $\mathrm{in}(U_{n+1}) \geq h(\mathrm{in}(U_n)) > A + \epsilon$, contradiction. Hence we conclude that $A = 1$. ■

The above result is what we need to make systematic the notion that the domains U_n get larger and larger, approaching the whole disk. Note that each U_n is also a covering space of $U_0 = U$ with the covering map c_n given by composing the maps

$$U_n \xrightarrow{p_n} U_{n-1} \xrightarrow{p_{n-1}} \cdots U_1 \xrightarrow{p_1} U_0,$$

i.e. $c_n = p_1 \circ \ldots \circ p_n$. Let $f_0 : (\tilde{U}, \tilde{0}) \to (U_0, 0)$ be the abstract universal covering map taking a fixed base point $\tilde{0}$ to 0. By standard covering space theory, the universal covering map f_0 factors through each c_n by covering maps, i.e. there exist covering maps $f_n : (\tilde{U}, \tilde{0}) \to (U_n, 0)$ such that $c_n \circ f_n = f_0$.

Now since the sequence $\{c_n\}$ is uniformly bounded (each function maps to $\mathbb{D}$), we can extract a subsequence converging uniformly on compact subsets to a holomorphic function $c : \mathbb{D} \to \mathbb{C}$. We can then extract a subsequence of the corresponding subsequence of the f_n such that f_n converges compactly to a holomorphic function $f : (\tilde{U}, 0) \to (\mathbb{C}, 0)$. Note that $f_0 = c \circ f$, i.e. the universal covering map f_0 factors through f. We will ultimately show that f is an isomorphism to its image, which is $\mathbb{D}$. Towards this goal we have

Proposition 2.2.6. *The map f is open and its image is $\mathbb{D}$.*

Proof. Now we want to show that each $z \in \mathbb{D}$ is in $f(\tilde{U})$. Choose R such that $1 > R > |z|$. By Proposition 2.2.5, there is some N such that if $n \geq N$, then $U_n \supset \mathbb{D}_R$. Now let $V \subset \tilde{U}$ be the connected component of $f_N^{-1}(D_R)$ containing $\tilde{0}$. The map $f_N : V \to D_R$ is a covering map, and hence an isomorphism, since $\mathbb{D}_R$ is simply connected. Choose some R' with $R > R' > |z|$, and let $K = V \cap f_N^{-1}(\overline{\mathbb{D}_{R'}})$, which is compact. For all $n \geq N$, we have $f_n^{-1}(\overline{D_{R'}}) \subset K$, by the Schwarz lemma applied to $p_n \circ p_{n+1} \circ \cdots \circ p_N$. Hence we can choose a sequence $\{w_n\}$ with $w_n \in f_n^{-1}(z)$ contained in the compact set K. The sequence has a limit point w, satisfying $f(w) = z$. So $\mathbb{D} \subset f(\tilde{U})$.

In particular, f is non-constant, so it is open by the Open Mapping Theorem for holomorphic functions. By the construction of f, we have $f(\tilde{U}) \subset \overline{\mathbb{D}}$. Since f is an open map, it follows that $f(\tilde{U}) = \mathbb{D}$, as desired. ■

Lemma 2.2.7. *If an open, surjective map $g : V \to Y$ can be post-composed with a continuous map $\pi : Y \to Z$, such that $p_0 = \pi \circ g : V \to Z$ is a covering map, then g is itself a covering map.*

Proof. Let $y \in Y$. We will construct an evenly covered neighborhood of y. Let N be an evenly covered neighborhood of $\pi(y)$ with respect to the map p_0. Let N_y be the connected component of the open set $\pi^{-1}(N)$ that contains y. Let W be a connected component of $g^{-1}(N_y)$. We claim that g maps W homeomorphically to N_y, which will be sufficient to verify that g is a covering map.

Towards this claim, we first show that W is a connected component of $p_0^{-1}(N)$. By assumption W is connected, and it is contained in $p_0^{-1}(N)$, so there is some connected component W' of $p_0^{-1}(N)$ that contains W. Then $g(W')$ is also connected, and it is contained in $\pi^{-1}(N)$. Hence $g(W')$ is contained in N_y. It follows that $W' \subset W$, and hence $W' = W$.

Now since N is evenly covered with respect to p_0, we see that p_0 maps W homeomorphically to N. Since $p_0 = g \circ \pi$, it follows that g maps W injectively into N_y. The restricted map is open, since g is. Hence to complete the proof of the claim, all that remains is to show that $g(W) = N_y$. Since g is an open map, $g(W)$ is open, so if we can show that $g(W)$ is also closed, then we will be done, since N_y is connected. Note that since $p_0|_W$ is a homeomorphism, it has an inverse $q : N \to W$. Now consider the map $h = q \circ \pi : Y \to V$. Note that $h \circ g$ is the identity on V, hence $h^{-1}(W) = g(W)$, and, by continuity of h, it then follows that $g(W)$ is closed, as desired.

■

We can apply the lemma to the map f, since $c \circ f$ is the covering map f_0. So f must be a covering map, and since f maps onto the simply connected space $\mathbb{D}$, it must in fact be a conformal isomorphism. Thus $\tilde{U} \cong \mathbb{D}$, completing the proofs of Proposition 2.2.3 and Theorem 2.2.1 (Uniformization of plane domains).

■

2.2.3 Consequences of Uniformization

Suppose that $p : \mathbb{D} \to U$ is a universal covering map. By the above results, we will always have such a covering map when U is a connected proper subset of $\mathbb{C}$. We will show that this map allows us to pass the Poincaré metric on $\mathbb{D}$ to a metric on U. Given a point $z \in U$, choose a $z' \in \mathbb{D}$ in the fiber of z with respect to the map p. We can then pushforward the Poincaré metric ds at z' to a metric at z. We now show that this definition does not depend on the choice of point z' in the fiber over z. Let z'' be some other point in the fiber. Since every universal cover is regular, there is a deck transformation of p taking z' to z''. This map is a conformal automorphism, and hence the conformally invariant metric ds is unchanged under this deck transformation.

With the above construction, we get a metric ds_U on U, such that p is a *local isometry*, mapping any sufficiently small neighborhood to its image by an isometry. In fact, every piecewise-smooth path γ maps to smooth map $p \circ \gamma$ of the same hyperbolic length, and it follows that if x, y are sufficiently close then $d_{\mathbb{D}}(x, y) = d_U(f(x), f(y))$. But p need not be a *global isometry*, since, for instance, if a fiber has two or more points, then the Poincaré distance between these two points in $\mathbb{D}$ is positive, but their images in U are the same, hence the distance between the images is 0.

The metric ds_U is also conformally invariant, since any conformal automorphism of U lifts to an automorphism of $\mathbb{D}$. Because of this property we will also refer to these metrics ds_U as Poincaré metrics, and we will call such spaces U *hyperbolic surfaces*, since the metric ds_U gives rise to a hyperbolic geometry similar to the hyperbolic geometry on $\mathbb{D}$ and $\mathbb{H}$ discussed in previous sections.

2.3 Properties of Hyperbolic Surfaces

We now have a much larger class of hyperbolic surfaces to study. In this section we investigate some of their properties, which will be needed to develop the basic theory of Julia and Fatou sets. Given a hyperbolic surface U let

$$B(z_0, r) := \{z : d_U(z_0, z) \leq r\},$$

be the ball of radius r about z_0 with respect to the Poincaré metric on U.

Proposition 2.3.1. *Let U be a hyperbolic surface and let $z_0 \in U$. The ball $B(z_0, r)$ is compact for any $r > 0$.*

Proof. Since $B(0, r)$ is evidently closed, it suffices to show that it is contained in a compact set.

Suppose first that $U = \mathbb{D}$. By composing with an isometry, we can assume that $z_0 = 0$. Integrating the metric given in Proposition 2.1.8 gives

$$d_U(z_0, z) = \frac{1}{2} \log \frac{1 + |z|}{1 - |z|}.$$

It follows that $B(0, r) \subset \mathbb{D}_R$ for some $R < 1$, and hence $B(0, r)$ is contained in the compact set $\overline{\mathbb{D}}_R$.

For the general case, consider the covering map $p : \mathbb{D} \to U$. We can assume that 0 is a point in $p^{-1}(z_0)$. Note that $B(z_0, r) \subset p(B(0, r))$, since p is a local isometry. By the previous paragraph, $B(0, r)$ is contained in a compact set K, and hence $B(z_0, r)$ is contained in the compact set $p(K)$. ∎

Proposition 2.3.2. *Every hyperbolic surface U is contained in the union of a nested sequences of compact subsets $K_1 \subset K_2 \subset \cdots$ of U.*

Proof. By Proposition 2.3.1 we can choose a basepoint $z_0 \in U$ and take $K_n = B(z_0, n)$. Since every point $z \in U$ has finite Poincaré distance from z_0, it follows that $U = \bigcup_n K_n$. ∎

We next develop an important tool for studying families of maps. Recall that a collection of holomorphic maps $\mathcal{F} \subset \{f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}\}$ is said to be a *normal family* if every sequence of maps in $\mathcal{F}$ has a subsequence that converges locally uniformly to a holomorphic map.

Theorem 2.3.3 (Montel)**.** *Let S be a Riemann surface and $\mathcal{F}$ a family of holomorphic maps $S \to \widehat{\mathbb{C}}$, with the property that there are three distinct points a, b, c such that $f(S) \subset \widehat{\mathbb{C}} \backslash \{a, b, c\}$ for all $f \in \mathcal{F}$. Then the family $\mathcal{F}$ is normal.*

Proof. We can assume that S is some small open subset U of the plane (normality is a local property, since we can enumerate a basis for the topology of S, and then diagonalize to get convergent subsequences). By composing with a Möbius transformation, we can also assume that $\{a, b, c\} = \{0, 1, \infty\}$. By Theorem 2.2.2, there exists a covering map $p : \mathbb{D} \to \mathbb{C} \backslash \{0, 1\}$. Each map $f \in \mathcal{F}$ lifts to a map $\tilde{f} : U \to \mathbb{D}$. Note that family of holomorphic maps $\{\tilde{f}\}$ is bounded, and hence normal (this follows from the Arzelà-Ascoli Theorem, where we use the local boundedness of the family and Cauchy's formula for the derivative to get equicontinuity). For any $g = \lim \tilde{f}_n$ a limit of a sequence in $\{\tilde{f}\}$, the image $g(U)$ may contain points in $\partial\mathbb{D}$. By the way that the covering map $p : \mathbb{D} \to \mathbb{C} \backslash \{0, 1\}$ was constructed, it extends to a map $P : \overline{\mathbb{D}} \to \widehat{\mathbb{C}}$. Then the family $\mathcal{F} = \{p \circ \tilde{f}\}$ is also normal, since if a sequence $\{\tilde{f}_n\}$ converges to g, then the corresponding sequence $\{p \circ \tilde{f}_n\}$ in $\mathcal{F}$ converges to $P \circ g$. ■

The above theorem was originally proved by Paul Montel (see [Montel, 1912]), and it was instrumental in early work on complex dynamics.

Next we prove a result concerning the Poincaré metric near the boundary of an embedded surface.

Proposition 2.3.4. *Let $U \subset \widehat{\mathbb{C}}$ be a hyperbolic surface, and let $z_1, z_2, \ldots$ be a sequence of points converging to a boundary point $\hat{z} \in \partial U$. Then for a given $r > 0$, the closed balls $B(z_n, r)$ converge uniformly to $\hat{z}$ as $n \to \infty$.*

Proof. The strategy is to think of the $B(z_n, r)$ as images of a fixed ball by a sequence of universal covering maps, and then apply Theorem 2.3.3 (Montel).

By composing with a Möbius transformation, we can assume that $U \subset \mathbb{C} \backslash \{0, 1\}$. Let $p_n : \mathbb{D}_n \to \mathbb{C} \backslash \{0, 1\}$ be the universal covering map taking 0 to z_n. Note that $p_n(B(0, r)) = B(z_n, r)$, since p_n is a local isometry. By Montel's theorem, there is a subsequence of $p_n|_{B(0,r)}$ that converges locally uniformly to a map $p : B(0, r) \to \mathbb{C} \backslash \{0, 1\}$. We claim that the image $V = p(B(0, r))$ is actually the single point $\hat{z}$. Note V must contain $\hat{z}$. If V is not just a single point, then it must be open, and hence intersect U. On the other hand, the balls $B(z_n, r)$ eventually escape any closed ball about the origin, and these balls exhaust U (Proposition 2.3.2). Thus V does not intersect U, and we conclude $V = \{\hat{z}\}$.

We have shown that there exists a subsequence of the $B(z_n, r)$ converging locally uniformly to $\hat{z}$. To get the result for the whole sequence we first note that the diameters, in the spherical metric on $\widehat{\mathbb{C}}$, of the $B(z_n, r)$ tend to 0, since otherwise there would be a subsequence of $B(z_n, r)$ with spherical diameters $\geq \delta$. We could then apply Montel's theorem as in the previous paragraph to get a subsequence converging locally uniformly to a constant map, which is impossible. Since $z_n \to \hat{z}$, it then follows that $B(z_n, r)$ converges uniformly to $\hat{z}$, as desired.

■

2.4 Classification of maps on hyperbolic surfaces

2.4.1 Hyperbolic Schwarz Lemma

We now use Lemma 2.1.3 (Schwarz) and hyperbolic isometries to show that holomorphic maps on the disk are distance non-decreasing.

Lemma 2.4.1 (Hyperbolic Schwarz Lemma). *Suppose that $f : \mathbb{D} \to \mathbb{D}$. Then for any two distinct points $p_1, p_2 \in \mathbb{D}$, we have $d_{\mathbb{D}}(p_1, p_2) \geq d_{\mathbb{D}}(f(p_1), f(p_2))$, with equality iff f is a conformal isomorphism.*

Proof. Recall that the Poincaré metric gives an inner product on each tangent space, and hence a norm, which we will denote $\|v\|$, on each tangent space. The map f gives rise to a linear map on tangent space $Df_p : T\mathbb{D}_p \to T\mathbb{D}_{f(p)}$ for each p. We define the norm of each such linear map to be $\|Df_p\| := \|Df_p(v)\|/\|v\|$, which is evidently independent of the choice of tangent vector v. Note that if f is an isomorphism, then it is a hyperbolic isometry, hence $\|Df_p\| = 1$ for all p. If f has a fixed point at p, then $\|Df_p\| = |f'(p)|$, since this is the factor by which f scales a tangent vector. If $p = 0$, then by Lemma 2.1.3 (Schwarz), $\|Df_0\| = |f'(0)| \leq 1$, with equality iff f is an isomorphism. Now composing with automorphisms, we see that $\|Df_p\| \leq 1$ for all p, regardless of whether f has a fixed point, and equality holds iff f is an isomorphism.

Now suppose p_1, p_2 are joined by some geodesic segment γ, thought of as a map $[0, 1] \to \mathbb{D}$. Then γ is contained in some compact subset $K \subset \mathbb{D}$. There is some constant c such that $\|Df_p\| \leq c$ for all $p \in K$. If f is an isomorphism we can choose $c = 1$, and otherwise we can choose $c < 1$. Now

$$d_{\mathbb{D}}(p_1, p_2) = \int_0^1 \|\gamma'(t)\| \geq \int_0^1 c^{-1} \|Df_{\gamma(t)}(\gamma'(t))\| dt \geq c^{-1} \cdot d_{\mathbb{D}}(f(p_1), f(p_2)).$$

(In the expression $\|\gamma'(t)\|$, we are thinking of $\gamma'(t)$ as the image of the unit length tangent vector under the map between tangent spaces $T[0,1]_t \to T\mathbb{D}_{\gamma(t)}$.) It follows that $d_{\mathbb{D}}(p_1, p_2) \geq d_{\mathbb{D}}(f(p_1), f(p_2))$, with equality iff f is an isomorphism.

■

The next theorem extends the above result to maps between general hyperbolic surfaces.

Theorem 2.4.2 (Schwarz-Pick). *Let $f : S \to S'$ be a map between hyperbolic surfaces, considered along with their Poincaré metrics. Then one of the following possibilities holds:*

1. *f is a conformal automorphism and isometry.*
2. *f is a covering map, but is not injective. In this case f is distance non-increasing, and a local isometry, but not a global isometry.*
3. *f strictly decreases all non-zero distances.*

Proof. We consider the lift of f to a map $\tilde{f} : \mathbb{D} \to \mathbb{D}$ from the universal cover of S to the universal cover of S'. By the above Lemma 2.4.1, it follows that $\tilde{f}$ is either a conformal automorphism or strictly decreases all distances. In the first case, f is either itself a conformal automorphism, or a covering map. In the second case, $\tilde{f}$ decreases the hyperbolic length of all paths, so it follows that f decreases all non-zero distances. ■

2.4.2 Classification

We apply the theorem of the previous subsection to the case of a self-map $f : S \to S$. These are the maps that are of interest in dynamics, since we can study the iterates $f^{\circ n}$ of f. The precise classification of such maps that we get will be essential to our proof of Theorem 2.6.1 (Five Possibilities).

Theorem 2.4.3 (Classification)**.** *For any holomorphic map $f : S \to S$ of hyperbolic surfaces, exactly one of the following four possibilities holds:*

1. (Attracting) *f has a fixed point p, contained in a neighborhood N such that all orbits $\{f^{\circ n}(z)\}$ of points $z \in N$ converge to p.*

2. (Escape) *Every orbit of f eventually escapes any compact subset $K \subset S$.*

3. (Finite Order) *Some iterate $f^{\circ n}$ is the identity and every point of S is periodic.*

4. (Irrational Rotation) *(S, f) is a* rotation domain*. That is, S is conformally isomorphic to a disk $\mathbb{D}$, punctured disk $\mathbb{D}^*$, or annulus $A_r = \{z : 1 < |z| < r\}$; and f corresponds to an irrational rotation $z \mapsto e^{2\pi i\theta}$, with $\theta \notin \mathbb{Q}$.*

Proof. It is immediate that no two of the above cases can hold at the same time. We apply Theorem 2.4.2 (Schwarz-Pick) to f.

Distance-Decreasing Case. Suppose that f is not a local isometry. By Schwarz-Pick, f must strictly decrease all distances. If every orbit eventually escapes any compact subset $K \subset S$ then we are in the *Escape* case.

So suppose there is some $z_0 \in S$ such that the sequence $\{z_n = f^{\circ n}(z_0)\}_{n \geq 0}$ visits some compact subset $L \subset S$ infinitely often. Let K be a compact neighborhood of $L \cup f(L)$. Since $d_S(f(z), f(w)) < d_S(z, w)$ for all $z, w \in S$, there is some constant $c_K < 1$ such that for $z, w \in K$, we have $d_S(f(z), f(w)) < c_k d_S(z, w)$. For any $z_m \in L$, we have $d_S(z_{m+2}, z_{m+1}) < c_K \cdot d_S(z_{m+1}, z_m)$. Since infinitely many of the z_m lie within K, it follows that $\lim_{n\to\infty} d_S(z_{n+1}, z_n) = 0$. Hence the sequence $\{z_n\}$ converges to a point p, which must be a fixed point of f, by continuity. Now let $B_r = \{z : d_S(p, z) < r\} \subset K$ be a small ball around p. For any $z \in B_r$, we have $d_S(p, f^{\circ n}(z)) < c_K^n \cdot d_S(p, z) < c_K^n r$. Hence the orbit of z converges to p. It follows that f belongs to the *Attracting* case.

Distance-Preserving Case. Now suppose that f is a local isometry. Suppose first that S is simply connected, so we can assume that $S = \mathbb{D}$. By Schwarz-Pick f is a covering map, so it must be an automorphism of $\mathbb{D}$, i.e. a Möbius transformation. If f has a fixed point in $\mathbb{D}$, then by the Schwarz lemma, f is a rotation, so we are either in the *Finite Order* or *Irrational Rotation* case, depending on whether the rotational angle is rational or irrational. If f does not have a fixed point in $\mathbb{D}$, then its extension to $\overline{\mathbb{D}}$ has one or more fixed points on $\partial\mathbb{D}$, in which case we see that orbits all converge to the boundary, which implies that we are in the *Escape* case.

Now suppose that S is not simply connected. If $f^{\circ k}$ is the identity for some k, then we are in the *Finite Order* case, so assume that this does not happen. Let $\phi : (\mathbb{D}, 0) \to (S, z_0)$ be the universal covering map, where z_0 is some fixed basepoint, and let $\mathcal{G}$ be the group of deck transformations of this covering. The map f lifts via ϕ to a map $F : \mathbb{D} \to \mathbb{D}$ that must be a conformal automorphism (since f was a covering map, by Schwarz-Pick), i.e. a Möbius transformation. Let Γ be the group generated by F and the elements of $\mathcal{G}$.

Suppose Γ is a discrete topological group (see [Munkres, 2000] for the definition and properties of topological groups). Since no iterate of f is the identity, no iterate of F is an element of $\mathcal{G}$. It follows that orbits under F escape compact subsets of $\mathbb{D}$, hence iterates of f escape compact subsets of S. Thus we are in the *Escape* case.

Now suppose instead that Γ is not discrete. Consider the closure $\overline{\Gamma}$ in the group of conformal maps $\mathbb{D} \to \mathbb{D}$. Let Γ_0 be the connected component of $\overline{\Gamma}$ containing the origin. If $g \in \mathcal{G}$, then $FgF^{-1} \in \mathcal{G}$. Hence $\overline{\Gamma}$, as well as Γ_0, conjugates $\mathcal{G}$ to itself. For any $g \in \mathcal{G}$, the map $m : \Gamma_0 \to \mathcal{G}$ given by $h \mapsto hgh^{-1}$ is continuous, and hence its image is connected. Since $\mathcal{G}$ is discrete (being a group of deck transformations), the image of m is a single point. Since the image contains the identity, we have $hgh^{-1} = g$, i.e. h and g commute, for all $g \in \mathcal{G}, h \in \Gamma_0$.

Lemma 2.4.4. *If $M_1, M_2 : \mathbb{D} \to \mathbb{D}$ are conformal maps which commute with one another and M_1 is not the identity, then M_2 belongs to a one-parameter group generated by M_1, that is independent of M_2.*

Proof of Lemma. Recall that by Remark 2.1.5, M_1 has either one interior fixed point, one fixed point on $\partial\mathbb{D}$, or two fixed points on $\partial\mathbb{D}$.

Case 1. Suppose M_1 has one interior fixed point, which we can assume is at 0, so M_1 is a rotation $z \mapsto e^{i\theta}z$. Then $e^{i\theta}M_2(0) = M_1 \circ M_2(0) = M_2 \circ M_1(0) = M_2(0)$. Since M_1 is not the identity, $e^{i\theta} \neq 1$, and so $M_2(0) = 0$. Thus M_2 is also a rotation, and so belongs to the natural one-parameter group generated by M_1.

Case 2. Suppose M_1 has one boundary fixed point. Mapping the problem to $\mathbb{H}$, we can assume that (the analogue of) M_1 fixes ∞. Then $M_1(z) = z + \lambda i$ for some real $\lambda \neq 0$. Note that M_2 must also fix ∞, so $M_2(z) = z + \lambda' i$, which means M_2 is in the natural one-parameter group generated by M_1.

Case 3. Finally, suppose that M_1 has two boundary fixed points. Mapping the problem to $\mathbb{H}$, we can assume that $\infty, 0$ are both fixed points of M_1. Then $M_1(z) = \lambda z$, with $\lambda \in \mathbb{R}_+$. Since M_2 commutes with M_1, it must either interchange the fixed points, or fix both of them. In the former case, $M_2(z) = \lambda'/z$, but this does not actually commute with M_1. So M_2 must be a map of the form $z \mapsto \lambda' z$, with $\lambda' \in \mathbb{R}_+$, i.e. M_2 belongs to the natural one-paramater group generated by M_1. ■

Proof of Theorem 2.4.3 (continued). Now choose $h \in \Gamma_0$ that is not the identity. We showed that every element of $\mathcal{G}$ commutes with h, so $\mathcal{G}$ is a subgroup of a one-parameter group H_1 generated by h. Since $\mathcal{G}$ is both discrete and infinite it must be of the form $\{g^{\circ n}\}_{n=-\infty}^{\infty}$ for some $g \in H_1$. Thus the fundamental group of S is isomorphic to an infinite cyclic group generated by one element, i.e. the integers. So S is a doubly connected subset of $\widehat{\mathbb{C}}$, and hence it is isomorphic to either $\mathbb{C}^*, \mathbb{D}^*$, or an annulus A_r for some $r > 0$. The first case cannot happen, since the universal cover of $\mathbb{C}^*$ is $\mathbb{C}$, not $\mathbb{D}$ (the covering map is $z \mapsto \exp(z)$). Thus S is either $\mathbb{D}^*$ or A_r, and we are in the *Irrational Rotation* case. ■

2.5 Julia and Fatou sets

We now turn to dynamics of holomorphic maps on the Riemann sphere. We introduce the Fatou set, and complementary Julia set, which formalize the notion of sensitive dependence on initial conditions. In this section, we restrict our study to maps $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ (that are not identically ∞), although Fatou and Julia sets can be defined in more general settings. Any holomorphic map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is rational, i.e. a ratio of two polynomials p/q. (Proof sketch: The preimage of ∞ is an isolated set, and by composing with a Möbius transformation, we can assume $f(\infty) = \infty$. So we can think of f as a meromorphic function. Multiplying by polynomial factors, we can eliminate the poles, giving a proper map $\mathbb{C} \to \mathbb{C}$, which must be a polynomial. It follows that f must have been rational to start.) We define the *degree* of f to be the maximum of the degrees of p and q, where p and q are chosen so that they do not share a common factor. Any rational map of degree at least 1 is surjective, by the Fundamental Theorem of Algebra.

Since $\widehat{\mathbb{C}}$ is not a hyperbolic surface, one might think that the hyperbolic theory that we have developed in the preceding sections would be of little use. In fact, we often consider subsets of $\widehat{\mathbb{C}}$ that are hyperbolic, and the Poincaré metric is an important tool.

2.5.1 Basic Definitions

The definitions of the Fatou and Julia sets are based on normal families.

Definition 2.5.1. The *Fatou set* of a rational map f consists of all points $z \in \widehat{\mathbb{C}}$ for which there exists a neighborhood $N \ni z$ such that the family $\{(f|_N)^{\circ n}\}$ is normal. We will refer to such an N as a *neighborhood of normality*. The *Julia set* $J(f)$ is the complement of the Fatou set in $\widehat{\mathbb{C}}$.

Note that by its very definition, the Fatou set is open, while the Julia set is closed. Both of these sets are robust in the following sense.

Lemma 2.5.2 (Invariance)**.** *We say that a set S is* fully invariant *under f if $f(S) \subset S$, and $f^{-1}(S) \subset S$, i.e. z belongs to the image of S iff z belongs to S. Both Julia and Fatou sets are fully invariant.*

Proof. Suppose z is in the Fatou set, and let $N \ni z$ be a neighborhood of normality given by the definition. Consider $f(z)$, and let $f^{\circ n_j}$ be some sequence of iterates. Note that $f^{\circ n_j+1}|_N$ has a locally uniformly convergent subsequence, so $f^{\circ n_j}|_{f(N)}$ does also. Thus $f(N)$ is a neighborhood of normality for $f(z)$, and so $f(z)$ is in the Fatou set. Similarly, we see that if $z' \in f^{-1}(z)$, then $f^{-1}(N)$ is a neighborhood of normality for z'. It follows that the Fatou set is fully invariant, and since $J(f)$ is the complement of the Fatou set, it is also fully invariant. ■

When analyzing dynamical systems, the study of fixed points is often a good starting point. Note that if a rational map has infinitely many fixed points, then its fixed points have a limit point, since $\widehat{\mathbb{C}}$ is compact. It then follows that f must equal the identity. So unless f is the identity (in which case the dynamics are trivial), the map has isolated and finite fixed points. We now classify these fixed points based on the derivatives of f at the point.

Definition 2.5.3. Suppose that $z_0 \in \widehat{\mathbb{C}}$ is a fixed point of f. We can choose a local coordinate chart so that z_0 corresponds to the origin. We define the *multiplier* λ of z_0 (with respect to f) to be the derivative $f'(0)$, using the chosen coordinates.

To check that the multiplier is a well-defined concept, we need to verify that the derivative above does not depend on the particular choice of coordinates. This follows from the chain rule and the formula for the derivative of an inverse function. However the multiplier would not in general be well-defined at a point not fixed by f. On the other hand, the concepts of *critical points* and *critical values* are well-defined, since if the derivative is 0 in one chart, then it is 0 in all charts.

Definition 2.5.4. A point $z_0 \in \widehat{\mathbb{C}}$ is a *periodic point* of f if there exists some k for which z_0 is a fixed point of $f^{\circ k}$. The *multiplier* of such a periodic point of f is defined to be the multiplier of $f^{\circ k}$ at its fixed point z_0. If k is chosen to be minimal, then k is said to be the *period* of z_0.

Notice that a fixed point is a periodic point of period 1. The behavior of periodic points is very closely connected with the behavior of fixed points.

Lemma 2.5.5 (Iterate invariance)**.** *For any integer $k > 0$, the Fatou set of $f^{\circ k}$ coincides with the Fatou set of f, and $J(f^{\circ k}) = J(f)$.*

Proof. It is clear that the Fatou set of f is contained in that of $f^{\circ k}$. For the converse, let z be a point in the Fatou set of $f^{\circ k}$, with $N \ni z$ a neighborhood of normality. Let $\{f^{\circ n_j}\}$ be some sequence of iterates of f. Note that any sequence in the family $\{f^{\circ nk}|_N\}_{n\geq 1}$ has a locally uniformly convergent subsequence, and the same holds for each of the families $\{f^{\circ nk+1}|_N\}, \ldots, \{f^{\circ nk+(k-1)}|_N\}$. At least one of these k families, suppose it is $\{f^{\circ nk}|_N\}$, has infinitely many elements in common with $\{f^{\circ n_j}|_N\}$. These common elements form a sequence in the family $\{f^{\circ nk}|_N\}$, and hence the sequence has a locally uniformly convergent subsequence. It follows that $\{f^{\circ n_j}|_N\}$ also has a locally uniformly convergent subsequence, so z is also in the Fatou set, as desired. The result for the complementary Julia set follows immediately. ■

2.5.2 Fixed and periodic points

The multiplier of a fixed point z_0 strongly influences the dynamics near that point.

Definition 2.5.6. Suppose f is a rational map and that z_0 is a periodic point with multiplier λ.

- If $|\lambda| < 1$, then z_0 is said to be *attracting*. If $\lambda = 0$, then z_0 is *superattracting*. If $0 < |\lambda| < 1$, then z_0 is *geometrically attracting*.
- If $|\lambda| > 1$, then z_0 is *repelling*.
- If $\lambda^n = 1$ for some n, and f is not the identity, then z_0 is *parabolic*.
- If $|\lambda| = 1$ and $\lambda^n \neq 1$ for any n, then z_0 is *indifferent*.

Proposition 2.5.7. *Every attracting fixed point z_0 of f is in Fatou set. Furthermore the set A of all $z \in \widehat{\mathbb{C}}$ whose orbits converge to z_0 is an open subset of the Fatou set. This set is called the* basin of attraction *of z_0. The connected component of A containing z_0 is called the* immediate basin of attraction.

Proof. Choosing local coordinates, we can assume that $z_0 = 0$. Choose μ such that $|\lambda| < \mu < 1$. By Taylor's theorem, there is some small ball (in the Euclidean metric) B around z_0 such that $|f(z)| < \mu|z|$ for all $z \in B$. It then follows that the iterates of $f|_B$ converge uniformly to the constant map $z \mapsto z_0$, and hence B is a neighborhood of normality. Now suppose $z \in A$. Then for some k, $f^{\circ k}(z) \in B$. It follows that $N = (f^{\circ k})^{-1}(B)$ is a neighborhood of z contained in A. Thus A is open. The iterates of $f|_N$ converge uniformly to z_0, so A is contained in the Fatou set. ∎

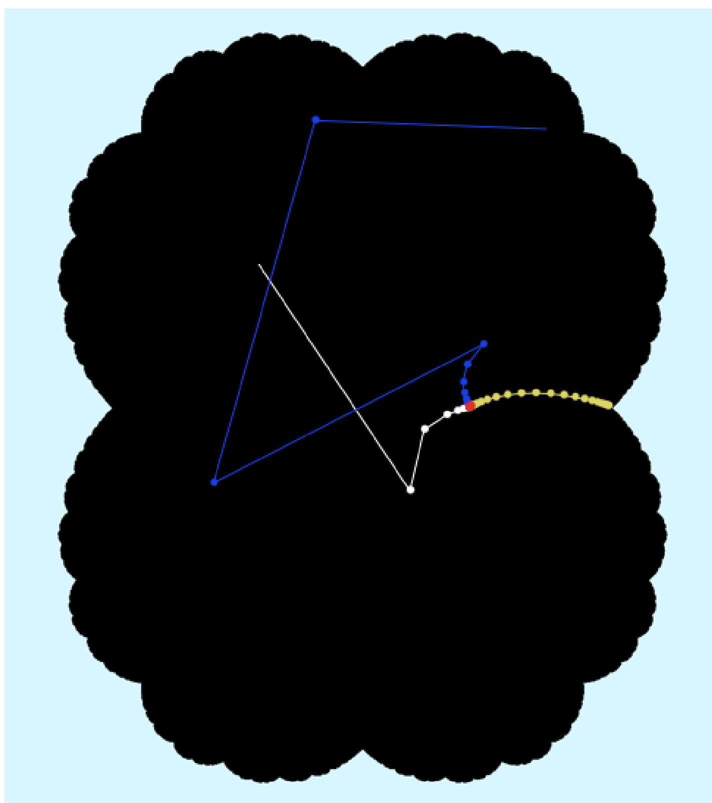

Figure 2.6: The basin for the attracting fixed point at 0 for the map $z \mapsto z^2 + 0.6z$. Several orbits are shown, all of which converge to the point colored in red. The boundary of the black region is the Julia set.

Proposition 2.5.8. *Every repelling fixed point z_0 of f is in the Julia set.*

Proof. Choosing local coordinates, we can assume that $z_0 = 0$. In the domain of these coordinates, the derivative of $f^{\circ k}$ at 0 is equal to λ^k. Since $|\lambda| > 1$, no subsequence of these derivatives will converge to a finite value. On the other hand, the derivatives of sequence of analytic functions converges to the derivative of the locally uniform limit of the functions, assuming such a limit exists. Hence the iterates of f cannot form a normal family on any neighborhood of z_0, and so z_0 is in the Julia set. ■

Proposition 2.5.9. *If f has degree at least two, then any parabolic fixed point z_0 is in the Julia set.*

Proof. As usual, by our choice of coordinates, we can assume that $z_0 = 0$. By taking an appropriate iterate of f, we an assume that the multiplier of the fixed point is equal to 1. On a small neighborhood of z_0, the Taylor series of f will have the form

$$f(z) = z + a_n z^n + \cdots ,$$

for some $a_n \neq 0$. Then

$$f^{\circ k}(z) = z + k a_n z^n + \cdots .$$

Hence, the nth derivative satisfies $(f^{\circ k})^n(0) = n! \cdot k \cdot a_n$, which goes to ∞ as $k \to \infty$. As in the proof of Proposition 2.5.8, we see that the iterates of f do not constitute a normal family on any neighborhood of z_0, so z_0 is in the Julia set. ■

These results easily generalize to periodic orbits. If z_0 is a periodic point, note that any point in the orbit of z_0 is also periodic, with the same multiplier and period as z_0. If f has an attracting periodic orbit of period m, then the attracting basin of the orbit is defined as the set A consisting of those points z for which $f^{\circ m}(z), f^{\circ 2m}(z), \ldots$ converges to a point in the periodic orbit.

2.5.3 The Julia Set

We now collect a few results concerning the structure of Julia sets that will be needed for the proof of the Five Possibilities Theorem.

Proposition 2.5.10 (Non-empty Julia set)**.** *If f has degree at least 2, then $J(f)$ is non-empty.*

Proof. Suppose for the sake of contradiction that the Fatou set of f is $\widehat{\mathbb{C}}$. If d is the degree of f, then $f^{\circ k}$ has degree dk. The sequence $\{f^{\circ n}\}_{n\geq 1}$ must have a subsequence $\{f^{\circ n_j}\}$ converging locally uniformly to a limit function g. For j sufficiently large, we have $\sup_z d(f^{\circ n_j}(z), g(z)) < \pi$, with the distance taken in the spherical metric (and π is the distance between antipodal points). Then $f^{\circ n_j}$ and g must have the same degree. On the other hand, g has some finite degree, while the degree of $f^{\circ n_j}$ is dn_j, which goes to infinity as $j \to \infty$, since $d \geq 2$. ■

Proposition 2.5.11 (Infinite Julia set)**.** *If f has degree at least 2, then $J(f)$ is an infinite set.*

Proof. Suppose for the sake of contradiction that $J(f)$ is some finite set. By Proposition 2.5.10, $n \geq 1$. Note that $J(f)$ is fully invariant (Lemma 2.5.2). Consider the finite directed graph G in which the vertices are the points of $J(f)$, and there is an edge from each $z \in J(f)$ to $f(z) \in J(f)$. Thus there is exactly one edge leaving each vertex, so the total number of edges in the graph is $|J(f)|$. For any $z \in J(f)$, the set $f^{-1}(z)$ is non-empty, since f is surjective, and any point in $f^{-1}(z)$ is in $J(f)$ since the set $J(f)$ is fully invariant under f. Hence there is at least one edge coming into each vertex of G. But the total number of edges is $|J(f)|$, the number of vertices, so it follows that there is exactly one edge coming into each vertex. This implies that each $z \in J(f)$ has exactly one preimage under f, and is hence a critical value of f. Thus every element of $J(f)$ is a critical point.

Now since $J(f)$ is finite, it must contain some periodic orbit. But since this periodic orbit consists of critical points, it is superattracting, and is hence in the Fatou set (Proposition 2.5.7), contradiction. ■

Proposition 2.5.12 (Iterated images)**.** *Let $z_0 \in J(f)$ be contained in a neighborhood $U \subset \widehat{\mathbb{C}}$, and let V be the union of the forward images of U, i.e. $V = \bigcup_{k=0}^{\infty} f^{\circ k}(U)$. Then $\widehat{\mathbb{C}} \backslash V$ contains at most 2 points.*

Proof. Suppose for the sake of contradiction, that $\widehat{\mathbb{C}} \backslash V$ contains at least 3 points. Note that $f(V) \subset V$. Thus by Theorem 2.3.3 (Montel), the iterates of $f|_V$ form a normal family. But this would imply that V is a subset of the Fatou set, and hence z_0 is in the Fatou set, contradiction. ■

Proposition 2.5.13 (Iterated preimages are dense)**.** *Let f be a map of degree at least 2, and let $z_0 \in J(f)$. Then the iterated preimages of z_0 are dense in $J(f)$.*

Proof. Let $z \in J(f)$ be contained in some neighborhood $U \subset \widehat{\mathbb{C}}$. Let $V = \bigcup_{k=0}^{\infty} f^{\circ k}(U)$. By Proposition 2.5.12, $V^c := \widehat{\mathbb{C}} \backslash V$ contains at most two points. We claim that these points are all in the Fatou set. Note that any preimage of a point in V^c is in V^c, since $f(V) \subset V$. Since V^c is finite, it follows that it contains some point w which equals some iterated preimage of itself, and so w is a periodic point. If V^c consists of exactly one or exactly two fixed points, then each has only one preimage (itself), so each is superattracting (since f has degree at least 2), hence in the Fatou set. If V^c consists of a single periodic orbit of period 2, then again each has only one preimage, hence each is both a critical value and a critical point, hence the orbit is superattracting and in the Fatou set. The last case is that V^c consists of a fixed point w_1 and a point w_2 with $f(w_2) = w_1$, but then w_2 does not have any preimages, contradicting the fact that f is surjective.

From the above, it follows that $z_0 \in V$, since z_0 is in the Julia set. Thus there is some $u \in U$ with $f^{\circ k}(u) = z_0$ for some k. It follows that the iterated preimages of z_0 are dense in $J(f)$.

■

Proposition 2.5.14 (No isolated Julia points)**.** *If f has degree at least 2, then the Julia set $J(f)$ contains no isolated points.*

Proof. By Proposition 2.5.11, $J(f)$ contains infinitely many points, and thus contains an accumulation point z_0. The iterated preimages of z_0 form a dense subset of $J(f)$, by Proposition 2.5.13. Suppose w_0 is a preimage, with $f^{\circ k}(w_0) = z_0$. Let U be any neighborhood of w_0. Then, by the Open Mapping Theorem, $f^{\circ k}(U)$ is an open set containing z_0. Since z_0 is an accumulation point of $J(f)$, $f^{\circ k}(U)$ contains a point $z' \in J(f)$ distinct from z_0. So U contains a w' with $f^{\circ k}(w') \in J(f)$, and $w' \neq w_0$. Since $J(f)$ is fully invariant, $w' \in J(f)$. It follows that w_0 is not an isolated point of $J(f)$. Thus the set of iterated preimages of z_0 contains no isolated points of $J(f)$.

Now given any $w \in J(f)$ contained in some neighborhood U, we can chose an iterated preimage w_0 of z_0 that is contained in U. Since w_0 is not isolated in $J(f)$, we see that U contains infinitely many points of $J(f)$. We conclude that w is not an isolated point of $J(f)$. ■

2.5.4 Fatou components

The main theorem of this paper concerns *Fatou components* of f, which are defined to be connected components of the Fatou set of f.

Proposition 2.5.15. *The image $f(U)$ of any Fatou component U is itself a Fatou component.*

Proof. The continuous image of any connected set is connected, so $f(U)$ is connected, and since it is part of the Fatou set, it must be contained in some Fatou component U'. We want to prove that $f(U) = U'$. It suffices to show that $f(U)$ is closed relative to the Fatou set. Consider the closure $\overline{U}$ of U in $\widehat{\mathbb{C}}$. As a closed subset of a compact set, $\overline{U}$ is compact. Hence its image $f(\overline{U})$ is also compact, hence closed. Notice that the boundary of U is contained in the Julia set (otherwise U would not be a connected component), and so the image of the boundary is also in the Julia set. Hence $f(\overline{U})$ consists of $f(U)$ along with some Julia set points. So $f(U)$ is equal to the intersection of the closed set $f(\overline{U})$ with the Fatou set of f, hence $f(U)$ is closed in the Fatou set, as desired. ∎

2.6 Five Possibilities Theorem

We now have the theory needed to state and prove the central result of this paper, a precise classification theorem describing the types of Fatou components that rational maps can exhibit.

Theorem 2.6.1 (Five Possibilities)**.** *Suppose $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational map of degree at least 2, and U is a connected component of the Fatou set such that $f(U) = U$. Then exactly one of the following holds:*

1. (Superattracting)*: U is an immediate basin of attraction for a superattracting fixed point.*

2. (Geometrically Attracting) *U is an immediate basin of attraction for geometrically attracting fixed point.*

3. (Parabolic) *U has a parabolic fixed point on its boundary to which all orbits in U converge.*

4. (Siegel Disk) U *is a* Siegel disk, *i.e. there is a conformal isomorphism* $U \to \mathbb{D}$ *that conjugates* f *to a rotation* $z \mapsto e^{2\pi i\theta z}$*, with* $\theta \in \mathbb{R}\backslash\mathbb{Q}$.

5. (Herman Ring) U *is a* Herman ring, *i.e. there is a conformal automorphism* $U \to A_r$*, where* $A_r = \{z : 1 < |z| < r\}$ *is some annulus, conjugating* f *to a rotation* $z \mapsto e^{2\pi i\theta z}$*, with* $\theta \in \mathbb{R}\backslash\mathbb{Q}$.

The main tools needed for the proof have already been developed in section 2.4. We will also need a few somewhat technical lemmas which we develop in the next two subsections. The proof will then be completed in subsection 2.6.3.

2.6.1 Boundary Fixed Points

Lemma 2.6.2 (Convergence to Boundary Fixed Point)**.** *Let* $f : \mathbb{C} \to \mathbb{C}$ *be rational of degree at least two, and* $U \subset \widehat{\mathbb{C}}$ *a hyperbolic surface with* $f(U) \subset U$*. Suppose that some orbit of* f *in* U *has no accumulation point in* U*. Then there is some point* $p \in \partial U$ *such that all orbits in* U *converge (in* $\overline{U}$*) to* p.

Proof. The strategy of the proof is as follows. All orbits behave approximately the same because of the non-increasing nature of f with respect to Poincaré distance. We choose an orbit without an accumulation point in U and construct a path in U following this orbit. Then the set of all the accumulation points of this path in ∂U will be connected and consist of fixed points of f. It follows that the set is just a single point, to which all orbits converge. Now we give the details.

Let $f : z_0 \mapsto z_1 \mapsto \cdots$ be an orbit without an accumulation point in U. Choose a continuous path $\gamma : [0, 1] \to U$ connecting z_0 to z_1. Then extend γ to $[0, \infty)$ by setting $\gamma(x + 1) = f(\gamma(x))$.

Let $\hat{z}$ be an accumulation point of $\gamma(x)$ as $x \to \infty$, and let $x_1, x_2, \ldots \in [0, 1)$, $x_n \to \infty$ be a sequence such that $\gamma(x_n)$ converges to $\hat{z} \in \partial U$. We claim that $\gamma(x_n + 1) = f(\gamma(x_n))$ also converges to $\hat{z}$. In fact $\gamma[0, 1])$ is compact, and hence contained in some ball $B(z_0, \delta) = \{z : d_U(z_0, z) \leq \delta\}$. Since $\gamma[n, n + 1] = f(\gamma[n - 1, n])$ and f is distance decreasing, $\gamma[x, x + 1] \subset B(\gamma(x), 2\delta)$, for any $x \in [0, \infty)$. Then by Proposition 2.3.4 the balls $B(\gamma(x_n), 2\delta)$ converge uniformly to $\hat{z}$. It follows that $\gamma[x_n, x_n + 1]$ converges to $\hat{z}$, and in particular $f(\gamma(x_n)) = \gamma(x_n + 1) \to \hat{z}$.

Since $\gamma(x_n) \to \hat{z}$ and $f(\gamma(x_n)) \to \hat{z}$, by continuity we have $f(\hat{z}) = \hat{z}$. Let A be the set of all accumulation points of $\gamma(x)$ as $x \to \infty$. We have that A consists of fixed points of f. It is easy to see that the $d_U(\gamma(x), \gamma(0)) \to \infty$ as $x \to \infty$, and it follows that $A \subset \partial U$.

Claim: The set of accumulation points of $\gamma(x)$ as $x \to \infty$ (or indeed of any continuous path as the parameter goes to infinity) is connected.

Proof: Recall that the intersection of a nested sequence of compact connected sets is again connected. The set of accumulation points that we are interested in is equal to $\bigcap_x \overline{p[x, \infty)}$. Each of the sets in the intersection is a closed subset of $\widehat{\mathbb{C}}$, hence compact, and the sets are nested and connected, so the claim follows.

Applying the claim, we see that A is a connected set, and it must be finite, since f has only finitely many fixed points. Hence A consists of just one point $\hat{p}$. Thus $\gamma(x)$ converges to $\hat{p}$ as $x \to \infty$, and in particular the orbit $z_n = \gamma(x_n)$ also converges to $\hat{p}$. For any $w \in U$, applying Proposition 2.3.4 to the ball $B(z_0, d_U(z_0, w))$, it follows easily that the orbit of w converges to $\hat{z}$. ■

2.6.2 Snail Lemma

Let f be a holomorphic function defined on some neighborhood of V of the origin, such that 0 is a fixed point of f with multiplier λ. We say that a path $\gamma : [0, \infty) \to V \backslash \{0\}$ *converges to the origin* if $\lim_{t\to\infty} \gamma(t) = 0$.

Lemma 2.6.3 (Snail). *In the above setting, if the path γ satisfies $f(\gamma(t)) = \gamma(t+1)$, then either $|\lambda| < 1$, or $\lambda = 1$.*

Proof. We give a sketch of the proof. Full details can be found in [Milnor, 2006] or [Carleson and Gamelin, 1993].

Note that the origin cannot be a repelling fixed point, so $|\lambda| \leq 1$. As $z = \gamma(t)$ approaches the origin, $f(z)$ is dominated by the linear term of its Taylor series. Since $\gamma(t+1) = f(\gamma(t))$, we get the asymptotic equality $\gamma(t+1) \sim \lambda\gamma(t)$ as $t \to \infty$. Assume that $|\lambda| = 1$, but $\lambda \neq 1$. Then, provided that γ has no self-intersections, the path resembles a spiral around the origin. A ray R from the origin will intersect this spiral many times. Consider a segment $S \subset R$ that intersects the spiral near 0 at the end points of S, and nowhere in between. Consider the region V bounded by S and the spiral. Note that V is open and simply connected, hence conformally isomorphic to the disk. See Figure 2.7.

It is possible to show that f maps V to itself. Also $0 \in V$ and $f(0) = 0$. We can then apply Lemma 2.1.3 (Schwarz) to $f : V \to V$, and since f is not an automorphism,

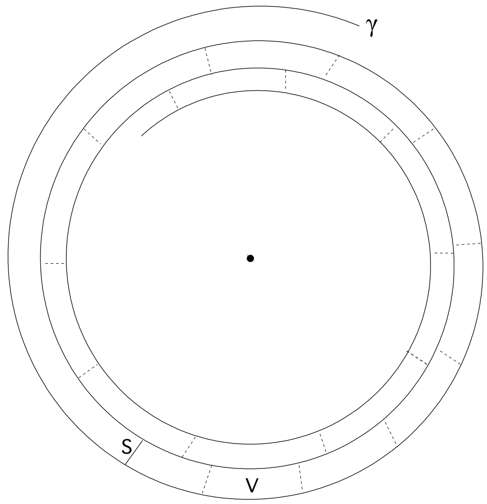

Figure 2.7: The path γ spiraling towards the origin.

we conclude that 0 is an attracting fixed point, which contradicts our assumption that $|\lambda| = 1$.

■

2.6.3 Proof of Five Possibilities Theorem

Proof of Theorem 2.6.1. Note that since the Julia set is infinite (Proposition 2.5.11) we can assume that $U \subset \mathbb{C}\backslash\{0, 1\}$, and so by Theorem 2.2.1 (Uniformization of plane domains), U is a hyperbolic surface. Most of the tools that we will need have already been developed in section 2.4. Consider the four cases of Theorem 2.4.3, as applied to $f|_U$.

Attracting. $f|_U$ must have either a superattracting or geometrically attracting fixed point, and since an immediate basin of attraction consists of an entire Fatou component, it follows that U must be *Superattracting* or *Geometrically Attracting.*

Escape. We can apply Lemma 2.6.2 (Convergence to a Boundary Fixed Point) to conclude that all orbits in U converge to some $p \in \partial U$. We claim that p is a parabolic fixed point. Choose some basepoint $z_0 \in U$, and a path $\gamma : [0, 1] \to U$ from $z_0 = \gamma(0)$ to $f(z_0) = \gamma(1)$. Extend γ to a map $[0, \infty) \to U$ by setting $\gamma(t + 1) = f(\gamma(t))$. This path must converge to the fixed point p. Applying Lemma 2.6.3 (Snail) gives that the multiplier λ of p satisfies either $0 < |\lambda| < 1$, or $\lambda = 1$. The first case cannot happen, since p is on the boundary of a Fatou component, and is hence in the Julia set, while all fixed points with $|\lambda| < 1$ are in the Julia set (see Proposition 2.5.7). In the second case, we get that p is a parabolic fixed point, as claimed. Since the orbits of all points in U converge to p, we see that U is *Parabolic*.

Finite Order. It easily follows that some iterate of f is the identity. But this cannot be, since f is rational, and its degree is assumed to be at least 2.

Irrational Rotation. If U is isomorphic to the disk, then U is a *Siegel Disk*, while if U is isomorphic to an annulus A_r, then U is a *Herman Ring*. The only remaining case is that U is isomorphic to $\mathbb{D}^*$. Then one component of the complement of U would consist of a single point, which would have to belong to the Fatou set, since the Julia set does not contain isolated points (see Proposition 2.5.14). But U is an entire Fatou component, so this cannot happen. ■

2.6.4 General Fatou components

Although Theorem 2.6.1 concerns a Fatou component U that is mapped to itself under f, the theorem can actually be applied in quite general contexts. A *periodic Fatou component* is a Fatou component such that $f^{\circ k}(U) = U$ for some k. For a periodic Fatou component, we can apply Theorem 2.6.1 to the function $f^{\circ k}$, which gives a classification of periodic Fatou components similar to the classification of Fatou components fixed by f.

It might also occur that U is an eventually periodic component, i.e. $f^{\circ k}(U)$ is a periodic Fatou component for some k. Note that $f^{\circ k}$, being a rational map, can be thought of as a branched covering map. Hence U is a branched cover of a periodic Fatou component.

A deep fact is that the above analysis actually applies to all Fatou components.

Theorem 2.6.4 (No wandering domains)**.** *Any Fatou component is eventually periodic.*

Proof. The proof uses the sophisticated technique of quasiconformal deformation, which is beyond the scope of this paper. See [Sullivan, 1985]. ■

Chapter 3

Structure of Fatou Components

The rest of this paper will be devoted to:

(i) Exploring the structure of each of the five types of Fatou components appearing in Theorem 2.6.1 (Five Possibilities).

(ii) Showing that for each case there is a rational map f of degree at least two which has that type of Fatou component.

Existence in the Geometrically Attracting, Superattracting, and Parabolic cases is obvious, since we can easily write down maps with the corresponding type of fixed points (see Figure 2.6 and Figure 3.1 for examples of the Geometrically Attracting and Parabolic cases, respectively). On the other hand, proving that there is a map with a Siegel disk and a map with a Herman ring is considerably more subtle, and it is harder still to give explicit examples.

We start by finding *model maps* for the Geometrically Attracting, Superattracting, and Parabolic cases that allow us to better understand the behavior of f near the fixed point.

Type of Fatou Component	Multiplier	Model Map
Geometrically Attracting	$0 < \lvert\lambda\rvert < 1$	$z \mapsto \lambda z$ (at 0)
Superattracting	$\lambda = 0$	$z \mapsto z^n$ (at 0)
Parabolic	$\lambda^k = 1$	$z \mapsto z + 1$ (at ∞)

Here the n in $z \mapsto z^n$ (Superattracting case) is equal to the degree of the critical fixed point, which is at least two. In the Geometrically Attracting and Superattracting cases, we will prove that f is conjugate to the model map on a neighborhood of the fixed point. In the Parabolic case, the connection between f and the model map is more subtle.

In the last two sections of this chapter, we demonstrate the existence of Siegel disks and Herman rings.

3.1 Geometrically Attracting basins

In this section, we consider a rational map f of degree at least two, with a fixed point, assumed to be at 0, of multiplier λ, with $0 < |\lambda| < 1$.

Our main result is that f is locally conjugate to the model linear map $z \mapsto \lambda z$. We then explore how far this conjugacy can be extended. Some of the results that we obtain will be used later in section 3.4 when we demonstrate the existence of Siegel disks.

3.1.1 Kœnigs Linearization

Recall that the basin of attraction $\mathcal{A}$ of 0 is the open set consisting of all points whose orbits under f converge to 0. It is clear that $f(\mathcal{A}) \subset \mathcal{A}$.

Theorem 3.1.1 (Kœnigs Linearization)**.** *If f is as above, then there is linearizing holomorphic map $\phi : \mathcal{A} \to \mathbb{C}$ such that the following diagram commutes:*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "f"] \arrow[d, "\phi"'] & \mathcal{A} \arrow[d, "\phi"] \\ \mathbb{C} \arrow[r, "z\mapsto\lambda z"'] & \mathbb{C} \end{tikzcd}$$

Proof. The diagram commutes iff $\lambda\phi(z) = \phi(f(z))$ for all $z \in \mathcal{A}$. The strategy is to define maps $\phi_0, \phi_1, \ldots$ so that

$$\lambda\phi_{n+1}(z) = \phi_n(f(z)). \tag{3.1}$$

Then if $\phi_n \to \phi$ locally uniformly as $n \to \infty$, we can take the limit of both sides of the above formula, giving $\lambda\phi(z) = \phi(f(z))$, as desired.

We will first define the functions ϕ_n in a small neighborhood U of 0. Choose c so that $|\lambda| < c < 1$. We take U to be a sufficiently small ball so that (i) there is a local analytic coordinate z on U; and (ii) $|f(z)| < c|z|$ for $z \in U$ (this is possible by Taylor's theorem).

Now define $\phi_0(z) = z$, and $\phi_{n+1}(z) = \phi_n(f(z))/\lambda$ for $n \geq 0$. Note that (3.1) holds, and $\phi_n(z) = f^{\circ n}(z)/\lambda^n$ for all n. Next we show that $\phi_n \to \phi$ locally uniformly on U. Consider a point z_0 and its orbit $f : z_0 \mapsto z_1 \mapsto \cdots$. We have

$$|\phi_{n+1}(z_0) - \phi_n(z_0)| = \left| \frac{f^{\circ(n+1)}(z_0)}{\lambda^{n+1}} - \frac{f^{\circ n}(z_0)}{\lambda^n} \right| = \frac{1}{|\lambda|^{n+1}} \left| z_{n+1} - \lambda z_n \right|.$$

By Taylor's theorem, $|f(z) - \lambda z| < Cz^2$ for some constant C. Since $z_{n+1} = f(z_n)$, and $z_n < c^n z_0$, we have $|z_{n+1} - \lambda z_n| < C(c^n z_0)^2$. With the above, this implies

$$|\phi_{n+1}(z_0) - \phi_n(z_0)| < \frac{z_0^2 \cdot C \cdot c^{2n}}{|\lambda|^{n+1}} = \frac{z_0^2 \cdot C}{|\lambda|} \cdot \left(c^2/|\lambda|\right)^n.$$

Now we could have initially chosen c such that $c^2 < |\lambda| < c < 1$. With this choice, the above difference goes to 0 exponentially in n and uniformly in the choice of z_0. Hence the sequence $\{\phi_n\}$ is uniformly Cauchy, and so it converges uniformly to an analytic function ϕ.

All that remains is to extend ϕ from the small neighborhood U to all of $\mathcal{A}$. We would like to just set $\phi(z) = \lim_{n\to\infty} f^{\circ n}(z)/\lambda^n$, but there may not be a local coordinate z defined on all of $\mathcal{A}$. Instead, to define $\phi_n(z)$, we start with z and follow its orbit until $z_k = f^{\circ k}(z)$ is in U. Then we set $\phi_n(z) = \lim_{n\to\infty} f^{\circ(n-k)}(z_k)/\lambda^n$. This defines an analytic function, since $\mathcal{A}$ converges locally uniformly to 0, and ϕ still satisfies (3.1). ■

Note that for the map ϕ constructed in the proof above, $\phi'(0) = 1$, and hence ϕ has a local inverse ψ_ϵ on some small neighborhood $\mathbb{D}_\epsilon$. The linearizing map ϕ is unique up to multiplication by a non-zero constant. In fact, for any other linearizing map $\tilde{\phi}$, the map $\phi^{-1} \circ \tilde{\phi}$ would have to commute with $z \mapsto \lambda z$, which implies that $\phi^{-1} \circ \tilde{\phi}$ is multiplication by a non-zero constant.

We now turn to a result concerning the extension of the map $\psi_\epsilon : \mathbb{D}_\epsilon \to \mathcal{A}$. Since the image is connected, we can define the range of ψ_ϵ to be $\mathcal{A}_0$, the immediate basin of attraction of 0.

Proposition 3.1.2 (Critical Point Obstruction)**.** *The map $\psi_\epsilon : \mathbb{D}_\epsilon \to \mathcal{A}_0$ extends to a map $\psi_r : \mathbb{D}_r \to \mathcal{A}_0$ on a maximal disk centered at the origin, and ψ_r extends homeomorphically to $\partial\mathbb{D}_r$. The image $\psi_r(\partial\mathbb{D}_r)$ contains a critical point of f.*

Proof. We first show that we cannot extend ψ_ϵ to all of $\mathbb{C}$. Such an extension ψ would be injective, since ϕ is an inverse, and thus ψ would be a conformal isomorphism to its image $U \subset \mathcal{A}_0 \subset \widehat{\mathbb{C}}$. But the only subsets of $\widehat{\mathbb{C}}$ conformally isomorphic to $\mathbb{C}$ are $\hat{\mathbb{C}} \backslash p$ for some p. This would imply that $|J(f)| \leq 1$, contradicting Proposition 2.5.11 (Infinite Julia set).

So let $r < \infty$ be the supremum over the radii of all disks to which we can extend ψ_ϵ. It follows easily that the supremum is achieved. Let $U \subset \mathcal{A}_0$ equal the image $\psi_r(\mathbb{D}_r)$.

We claim that $\overline{U} \subset \mathcal{A}_0$. Note that since ψ_r conjugates $z \mapsto \lambda z$ to f, we have

$$f(U) = \psi_r(\lambda \mathbb{D}_r) \subset \psi_r(\lambda \overline{\mathbb{D}}_r),$$

and $K := \overline{\mathbb{D}}_r$ is a compact set in $\mathcal{A}_0$. Now by continuity and the fact that K is closed,

$$f(\overline{U}) \subset \overline{f(U)} \subset K \subset \mathcal{A}_0.$$

Since $\overline{U}$ is mapped by f into the attracting basin, it must itself be in the basin, and since it is connected, $\overline{U} \subset \mathcal{A}_0$.

It follows from the above that we can extend ψ_r to the boundary $\partial \mathbb{D}_r$, which is mapped homeomorphically to ∂U. Now consider a point $z_0 \in \partial \mathbb{D}_r$. Its image $w_0 = \psi(z_0)$ lies in ∂U. We wish to extend ψ_r to a neighborhood of z_0. Note that $f(w_0) \in U$. If w_0 is not a critical point of f, we can locally find a branch h of the inverse f^{-1} mapping a neighborhood $W \subset U$ of $f(w_0)$ to a neighborhood V of w_0. Let $N = \lambda^{-1}\phi(W)$, which contains z_0. We can extend ψ_r to N by setting $\psi_r(z) = h(\psi_r(\lambda z))$.

Hence we see that if ∂U contains no critical points of f, then for each point $z_0 \in \partial \mathbb{D}_r$, we can extend ψ_r to a small neighborhood of z_0. Piecing together these extensions would give an extension ψ_R to some disk $\mathbb{D}_R$, with $R > r$, contradicting the maximality of r. We conclude that ∂U contains at least one critical point of f.

■

Remark 3.1.3. We can apply much of the above discussion to a map f with a *repelling* fixed point ($|\lambda| > 1$). In this case, f will locally have an inverse h near its fixed point, and h will have a geometrically attracting fixed point. However, the basin of a repelling fixed point cannot be sensibly defined.

3.2 Superattracting basins

We next consider a rational map f of degree at least two, with a fixed point, assumed to be at 0, with multiplier 0. Thus 0 is a critical point of f. If the critical point is of degree n, we can write

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + \cdots ,$$

holding in a small neighborhood of 0, where $a_n \neq 0$ and $n \geq 2$.

Note that any polynomial map f of degree at least two has a superattracting fixed point at ∞. This can easily be seen using the coordinate chart $\zeta = 1/z$. The degree of the critical point is equal to degree of the polynomial. The presence of this superattracting fixed point frequently distinguishes the study of the dynamics of polynomials as easier than the study of the dynamics of general rational maps.

3.2.1 Böttcher's Theorem

Theorem 3.2.1 (Böttcher)**.** *If f is as above, then there is some neighborhood U of 0 on which f is conjugate to the model map $z \mapsto z^n$. That is, there is an injective map $\phi : U \to \mathbb{C}$ such that the following diagram commutes.*

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\phi"'] & U \arrow[d, "\phi"] \\ \mathbb{C} \arrow[r, "z\mapsto z^n"'] & \mathbb{C} \end{tikzcd}$$

Proof. The technique is rather similar to that used to prove Theorem 3.1.1 (Koenigs Linearization). We construct a sequence of maps ϕ_n satisfying

$$[\phi_{n+1}(z)]^n = \phi_n(f(z)),$$

and then take limits to yield a map ϕ satisfying $[\phi(z)]^n = \phi(f(z))$, which is equivalent to the condition that the diagram in the statement commutes. As in the Koenigs Linearization proof, we have to show that the ϕ_n converge locally uniformly. There is an additional subtlety, because we would like to set

$$\phi_{n+1}(z) = \sqrt[n]{\phi_n(f(z))} = \sqrt[n^k]{f^{\circ k}(z)},$$

but the nth root is not in general defined as a holomorphic function. However we can naturally define the root in terms of the Taylor series for f at 0.

See [Carleson and Gamelin, 1993] for the details.

■

3.3 Parabolic petals

In this section we consider the behavior of a rational map f around a parabolic fixed point. We assume that the fixed point is at 0, and that the degree of f is at least two. We will also assume that $\lambda = 1$. This isn't terribly restrictive, since we can take an appropriate iterate of f such that this holds. Also, by Lemma 2.6.3 (Snail), the parabolic point on the boundary of a Parabolic Fatou component will always have multiplier $\lambda = 1$. Under these assumptions, we can write

$$f(z) = z + az^{n+1} + \cdots = z(1 + az^n + \cdots),$$

where $a \neq 0$, and $n \geq 1$.

The behavior turns out to be similar to the behavior of the map $z \mapsto z+1$ near its fixed point at ∞. This similarity justifies the use of the term "parabolic" to describe these fixed points (recall that by Remark 2.1.5, maps $\mathbb{H} \to \mathbb{H}$ with one boundary fixed point, such as $z \mapsto z + 1$, are called parabolic).

3.3.1 Attracting directions

Motivation:

First we perform some (non-rigorous) formal manipulation to provide motivation. We begin by "conjugating" f by the map $z \mapsto z^n$. This gives a map

$$g(z) = z(1 + az + \cdots)^n = z(1 + naz + \cdots).$$

Then conjugating g by $z \mapsto c/z$ (where c is to be chosen later) gives

$$F(z) = z\left(1 + \frac{nac}{z} + o\left(\frac{1}{|z|}\right)\right)^{-1} = z\left(1 - \frac{nac}{z} + o\left(\frac{1}{|z|}\right)\right),$$

where we have applied the generalized binomial theorem. Setting $c = -1/na$ gives

$$F(z) = z + 1 + o(1), \tag{3.2}$$

as $|z| \to \infty$. Formally, we have

$$F = \varphi \circ f \circ \varphi^{-1},$$

where

$$\varphi(z) = c/z^n$$

The function φ^{-1} is multi-valued, but we will ignore this issue for the moment. The parabolic point of f at 0 corresponds to the parabolic point of F at ∞. This map F will be easier to analyze, since the error term has a nicer form. If we consider z_0 with magnitude sufficiently large, it will turn out that the points of the orbit $\{z_k = F^{\circ k}(z_0)\}$ will satisfy $z_k \sim k$ as $k \to \infty$, i.e. the points of the orbit go to ∞ along the positive real axis $\mathbb{R}_+$. Translating these observations to the original map f will give orbits of f that converge to 0 along $\varphi^{-1}(\mathbb{R}_+)$, which consists of n rays in the directions of the n solutions of $v^n = c$.

Let v_0 be some solution to $v^n = c$, and for $j = 1, \ldots n-1$, let

$$v_j = e^{2j\pi i/n} v_0. \tag{3.3}$$

We denote these as *attracting directions*, and abuse notation a little by thinking of them as directions. Orbits along these directions will be attracted to 0. For $j = 0, \ldots, n-1$, let

$$w_j = e^{(2j+1)\pi i/n} v_0. \tag{3.4}$$

These will be called *repelling directions*. Note that $v_0, w_0, v_1, w_1, \ldots, v_{n-1}, w_{n-1}$ constitute $2n$ equally spaced directions about 0.

Theorem 3.3.1. *Let $f : z_0 \mapsto z_1 \mapsto \cdots$ be an orbit converging to 0, with $z_i \neq 0$ for each i. Then for some v_j, we have $z_k \sim v_j/\sqrt[n]{k}$ as $k \to \infty$ (i.e. the ratio of the two expressions approaches 1). In this case, we say that the orbit* converges to 0 in the direction *v_j.*

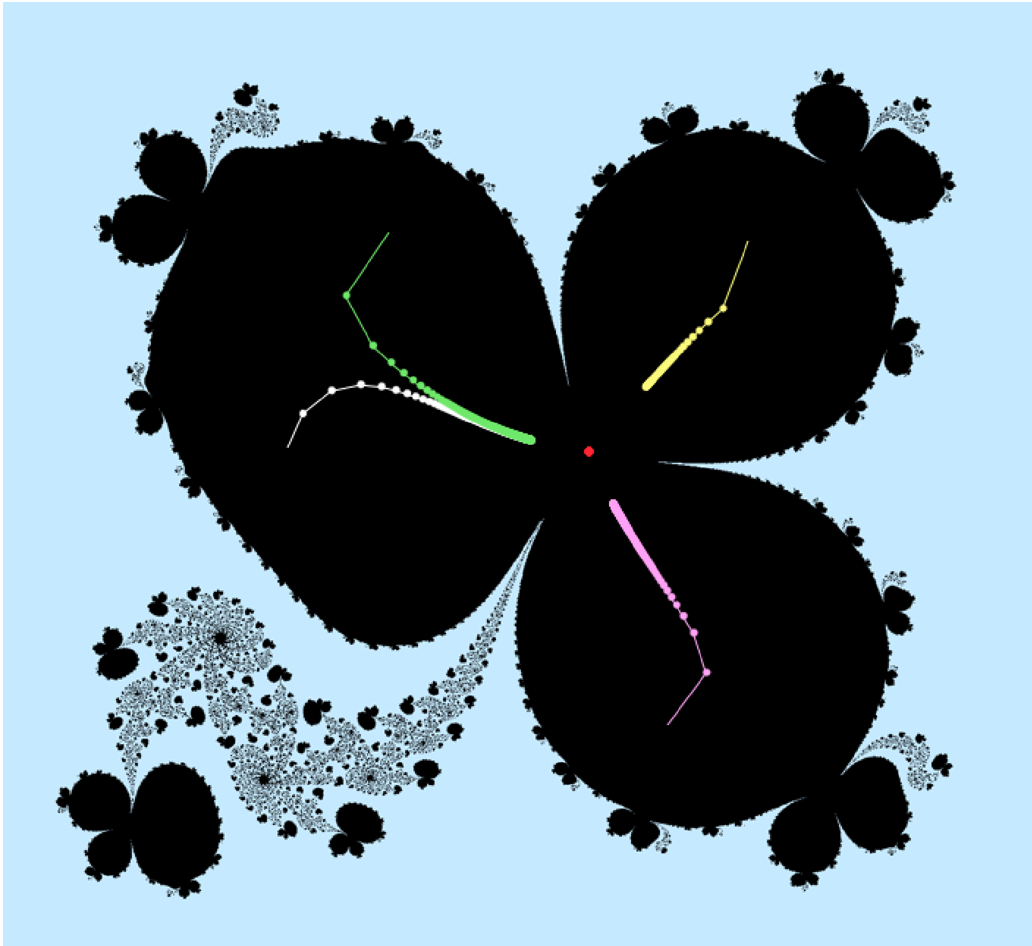

Figure 3.1: The map $z \mapsto z^5 + (0.8 + 0.4i)z^4 + z$ has a parabolic fixed point at 0. Several orbits are drawn, each of which converges to the fixed point (red) along one of the three attracting directions. The boundary of the black region is the Julia set.

Proof. The only problem with above analysis is that we cannot actually conjugate by φ, since $\varphi^{-1}(z) = \sqrt[n]{c/z}$ is multi-valued. To handle this issue, we restrict the domain of φ to various sectors around 0.

We now define the sectors. For $j = 0, \ldots, n-1$, let

$$A_j = \{re^{i\theta}v_j : r > 0, |\theta| < \pi/n\} \tag{3.5}$$

$$R_j = \{re^{i\theta}w_j : r > 0, |\theta| < \pi/n\}. \tag{3.6}$$

Each A_j has an attracting direction along its middle, while each R_j has a repelling direction along its middle.

Now observe that the map φ is injective on each of these sectors. In fact φ maps A_j biholomorphically to $\mathbb{C}\backslash\mathbb{R}_-$, while R_j is mapped biholomorphically to $\mathbb{C}\backslash\mathbb{R}_+$. Let ψ_j denote the inverse of $\varphi|_{A_j}$. Now we define

$$F_j = \varphi \circ f \circ \psi_j.$$

As in (3.2), we still have

$$F_j(z) = z + 1 + o(1), \text{ as } z \to \infty. \tag{3.7}$$

If $|z|$ is sufficiently large, then

$$\operatorname{Re} F_j(z) > \operatorname{Re} z + 1/2. \tag{3.8}$$

It follows that for large $R > 0$, the half-plane $\mathbb{H}_R = \{z \in \mathbb{C} : \operatorname{Re}(z) > R\}$ is mapped to itself, i.e. $F_j(\mathbb{H}_R) \subset \mathbb{H}_R$.

Now let $P_j(R) = \psi_j(\mathbb{H}_R) \subset A_j$; as a connected subset of a sector, this set has some resemblance to a petal of a flower. We claim that $f(P_j(R)) \subset P_j(R)$. Note that the rays along the w_k are mapped by φ to $\mathbb{R}_-$. Since $F_j(\mathbb{H}_R)$ does not intersect $\mathbb{R}_-$, it follows that $f(P_j(R))$ does not intersect any ray in the direction w_k. Also $P_j(R)$, and hence $f(P_j(R))$, are connected, so $f(P_j(R)) \subset A_l$ for some l. Now $P_j(R)$ contains the point rv_j for small r, and $f(rv_j) = z(1 + az^n + \cdots) = zw$, where $\arg(w) < \frac{\pi}{2n}$ (if $r \ll 0$). It follows that $f(rv_j) \in P_j(R)$, and hence $f(P_j(R)) \subset P_j(R)$, as claimed. It is also easy to see that $P_j(R) \to 0$ uniformly as $R \to \infty$.

Consider the orbit $f : z_0 \mapsto z_1 \mapsto \cdots$ in the statement of the theorem. Note that $\operatorname{Re} \varphi(z_k) > \operatorname{Re} \varphi(z_{k-1}) + 1/2$, by (3.8). If R is chosen as above, for sufficiently large k, $\varphi(z_k)$ will lie in $\mathbb{H}_R$. Hence z_k will be in some petal $P_j(R)$. Since each such petal is mapped to itself by f, the petal $P_j(R)$ eventually absorbs all elements of the orbit $z_0 \mapsto z_1 \mapsto \cdots$.

Let $\hat{z}_k = \varphi(z_k)$. By (3.7), we see that

$$\lim_{k\to\infty} \frac{\hat{z}_k}{k} = 1.$$

Writing $\hat{z}_k = c/z_k^n$ gives

$$1 = \lim \frac{c}{z_k^n \cdot k} = \lim \left(\frac{v_j}{z_k \sqrt[n]{k}} \right)^n .$$

Since we already know that (for $k \gg 0$) z_k is in the petal $P_j(R)$, which also contains complex numbers along the direction v_j, we can extract an appropriate nth root to get

$$\lim_{k \to \infty} \frac{v_j}{z_k \sqrt[n]{k}} = 1,$$

i.e. $z_k \sim v_j / \sqrt[n]{k}$. ■

Definition 3.3.2. Suppose f is a rational map with a parabolic fixed point z_0. Then a *attracting parabolic petal* for the direction v (with respect to the map f and the fixed point z_0) is an open set $P \subset \widehat{\mathbb{C}}$ such that

(i) An orbit $\{w_k\}$ of f converges to z_0 in the direction v iff $w_k \in P$ for all sufficiently large k.

(ii) $f(P) \subset P$.

The petals $P_j(R)$ in the proof of Theorem 3.3.1 are attracting parabolic petals. The proof also showed that the only directions along which orbits can approach 0 are the n equally spaced directions corresponding to $v_0, \ldots, v_{n-1}$ given in (3.3).

3.3.2 Repelling petals

We can define the map f^{-1} near 0. The vectors $w_0, \ldots, w_{n-1}$ (see (3.4)) are attracting directions for f^{-1}. The attracting parabolic petals of f^{-1} are called *repelling parabolic petals* of f.

3.4 Siegel disks

Recall that for f rational of degree at least two, a Fatou component U is a *Siegel disk* if there is a conformal isomorphism $h : U \to \mathbb{D}$ that conjugates f to a rotation $z \mapsto e^{2\pi i \theta z}$, with $\theta \in \mathbb{R} \backslash \mathbb{Q}$. The point $z_0 = h^{-1}(0)$ in U corresponding to 0 is a fixed point of f with multiplier $\lambda = e^{2\pi i \theta}$.

Definition 3.4.1. Let z_0 be an indifferent fixed point of f. If there does *not* exist a neighborhood U of z_0 with a conformal isomorphism $h : U \to \mathbb{D}$ conjugating f to a rotation, then we say that z_0 is a *Cremer point* of f.

Thus any indifferent fixed point z_0 either has a Siegel disk about it (the map is *linearizable* near z_0), or z_0 is a Cremer point. Clearly there are many maps with indifferent fixed points. Both Siegel and Cremer cases actually occur, but this is not at all obvious. Next we non-constructively demonstrate that Siegel disks exist.

3.4.1 Existence of Siegel disks

Theorem 3.4.2. *Let $q_\lambda(z) = z^2 + \lambda z$, for $\lambda = e^{2\pi i\theta}$, $\theta \in [0, 1)$. Note that q_λ has an indifferent fixed point at 0, with multiplier λ. The set of θ for which q_λ has a Siegel disk at z_0 has full Lebesgue measure in $[0, 1)$.*

Thus, in some sense, not only do Siegel disks exist, they are actually overwhelmingly common, at least for this family of quadratic polynomials. The proof will require several preliminary definitions and lemmas.

Definition 3.4.3 (Conformal radius)**.** Let $U \subset \widehat{\mathbb{C}}$ be a region conformally isomorphic to $\mathbb{D}$, and suppose that $0 \in U$. There is a unique $r > 0$ such that $g : \mathbb{D}_r \to U$ is a conformal isomorphism, with $g(0) = 0$ and $|g'(0)| = 1$. This value $\rho(U) := r$ will be called the *conformal radius* of U.

Now we consider the maps q_λ for $\lambda \in \overline{\mathbb{D}}\backslash\{0\}$. Each has a fixed point at 0 of multiplier λ. For $|\lambda| < 1$, the fixed point will be geometrically attracting, and hence there is *linearizing neighborhood* U of 0 (and a map $h : U \to \mathbb{D}$ conjugating f to $z \mapsto \lambda z$). For $|\lambda| = 1$, we do not yet know whether or not there is a linearizing neighborhood, except in the special circumstance when λ is a root of unity, in which case there is no such neighborhood since 0 is a parabolic fixed point in the Julia set. We now define a function $\mathrm{CR} : \overline{\mathbb{D}} \to \mathbb{R}$, mapping λ to the greatest conformal radius of a linearizing neighborhood about 0 of q_λ, i.e.

$$\mathrm{CR}(\lambda) = \begin{cases} 0 & \text{if } q_\lambda \text{ does not have a linearizing neighborhood about } 0 \\ \sup\{\rho(U) : U \text{ is a linearizing neighborhood about } 0 \text{ of } q_\lambda\} & \text{else.} \end{cases}$$

To prove Theorem 3.4.2, it will be sufficient to show that $\mathrm{CR}(\lambda) > 0$ for Lebesgue almost every $\lambda \in \partial\mathbb{D}$. Towards this end, we study the function CR on $\mathbb{D}\backslash\{0\}$, and examine the limiting behavior as $|\lambda| \to 1$. We will not be able to show that CR is continuous, but we will find a substitute condition.

Definition 3.4.4 (Semicontinuity)**.** A real-valued function u on a topological space is *upper semicontinuous* if

$$\limsup_{y \to y_0, y \neq y_0} u(y) \leq u(y_0).$$

In other words, the value of the function at a point should be at least what one would expect from nearby values if the function were continuous.

Proposition 3.4.5. *The function* CR *is bounded and upper semicontinuous on* $\overline{\mathbb{D}}$*.*

Proof. To see that it is bounded, note that if $|z| \geq 3$, then $|q_\lambda(z)| = |z(z+\lambda)| \geq 2|z|$, and so $q_\lambda^{\circ k}(z) \to \infty$. Thus any linearizing neighborhood is contained in $\mathbb{D}_3$, and $\mathrm{CR}(\lambda) \leq 3$.

For upper semicontinuity, suppose that $\limsup_{\lambda \mapsto \lambda_0} \mathrm{CR}(\lambda) = r$. Then there is a sequence of linearizing neighborhoods $U_n \subset \mathbb{D}_3$ with maps $h_n : (\mathbb{D}_{r_n}, 0) \to (U_n, 0)$ conjugating q_{λ_n} to $z \mapsto \lambda_n z$, and $r_n \to r$, $\lambda_n \to \lambda_0$. These maps form a normal family, so there is a subsequence converging locally uniformly to a map $h : \mathbb{D}_r \to \mathbb{D}_3$. The map h is a conformal automorphism to its image U and it conjugates q_{λ_0} on U to $z \mapsto \lambda_0 z$ on $\mathbb{D}_r$. Hence $\mathrm{CR}(\lambda_0) \geq r$, as desired. ■

We will need the following classical result concerning the radial limits of bounded analytic functions.

Lemma 3.4.6 (Riesz brothers)**.** *Let* $g : \mathbb{D} \to \mathbb{C}$ *be bounded and holomorphic. For any* $c_0 \in \mathbb{C}$*, the set*

$$E := \{\xi \in [0,1) : \lim_{r \to 1} g(re^{2\pi i \xi}) = c_0\}$$

has Lebesgue measure 0*.*

Proof of Lemma. Without loss of generalization, we can set $c_0 = 0$. Assume, for the sake of contradiction, that $\mu(E) > 0$. We can show that there is a subset $E' \subset \partial\mathbb{D}_r$, $r < 1$, such that $\mu(E') > 0$ and g is arbitrarily close to 0 on all of E'. Since g is bounded, this implies that we can choose r so that

$$\int_{\partial\mathbb{D}_r} \log|g(z)| dz < -N,$$

for any $N > 0$. But by Jensen's formula (see [Stein and Shakarchi, 2003]), the integral is a non-decreasing function of r, and hence bounded below, contradiction. ■

Proof of Theorem 3.4.2. We consider the function CR on the region $0 < |\lambda| < 1$. We claim that $\mathrm{CR}(\lambda) = |g(\lambda)|$ for some analytic function g. Let $\mathcal{A}_\lambda$ be the basin of attraction, with respect to q_λ, of the geometrically attracting fixed point 0.

By Theorem 3.1.1 and its proof, the linearizing Kœnigs map $\varphi_\lambda : \mathcal{A}_\lambda \to \mathbb{C}$ is given by

$$\varphi_\lambda(z) := \lim_{n\to\infty} q_\lambda^{\circ n}(z)/\lambda(z).$$

It depends holomorphically on each of λ, z separately.

We can assume that $\varphi_\lambda(0) = 0$ and $\varphi'_\lambda(0) = 1$. The inverse ψ_λ is defined locally on sufficiently small neighborhoods of U. By Proposition 3.1.2, there is some $r > 0$ such that we can extend ψ_λ as far as $\mathbb{D}_r$ but no farther, and ψ_λ extends homeomorphically to $\partial\mathbb{D}_r$, with $\psi_\lambda(\mathbb{D}_r)$ containing the (unique) critical point $-\lambda/2$ of f. Note that $\psi_\lambda : (\mathbb{D}_r, 0) \to (\psi_\lambda(\mathbb{D}_r), 0)$ is a conformal automorphism with $|\psi'_\lambda(0)| = 1$, and its image is a linearizing neighborhood for f at 0. It follows that $\mathrm{CR}(\lambda) \geq r$. The critical point of f serves as an obstruction to extending ψ_λ any farther, so in fact $\mathrm{CR}(\lambda) = r$. Note that $|\varphi_\lambda(-\lambda/2)| = r$. Thus we can take $g(\lambda) = \varphi_\lambda(-\lambda/2)$, which is holomorphic.

Observe that $\mathrm{CR}(0) = 0$, and since CR is upper semicontinuous, it follows that $\lim_{\lambda\to 0} |g(z)| = \lim_{z\to 0} \mathrm{CR}(z) = 0$. Since CR, and hence g, is bounded, we can extend g to a (bounded) analytic function on all of $\mathbb{D}$, by the Riemann Removable Singularity Theorem.

Now suppose that $\lambda = e^{2\pi i\theta}$ is such that q_λ does *not* have a Siegel disk at 0. Then $\mathrm{CR}(\lambda) = 0$. Hence, by upper semicontinuity,

$$0 = \mathrm{CR}(\lambda) \geq \limsup_{r\to 1} \mathrm{CR}(re^{2\pi i\theta}) = \limsup_{r\to 1} |g(re^{2\pi i\theta})|,$$

and so it follows that $\lim_{r\to 1} g(re^{2\pi i\theta}) = 0$. Then applying Lemma 3.4.6 to g with $c_0 = 0$, we see that the set of such θ with this property has Lebesgue measure 0.

We conclude that measure of the set of $\theta \in [0, 1)$ for which q_λ has a Siegel disk at 0 is exactly 1, as desired. ■

3.4.2 Cremer points

The theory of Cremer points and Herman disks is subtle, and still not completely resolved. Much depends on the value of the multiplier $\lambda = e^{2\pi i\theta}$, in particular, on how well the angle θ can be approximated by rational numbers. Yet it is still unknown if the multiplier completely determines whether a fixed point has a Siegel disk or is a Cremer point. We cite one result which gives a relatively easy way of demonstrating that Cremer points do in fact exist.

Theorem 3.4.7 (Cremer)**.** *Suppose $|\lambda| = 1$ and $\lambda^n \neq 1$ for all n. Let f be a rational map of degree $d \geq 2$, and*

$$\liminf |\lambda^n - 1|^{1/d^n} = 0.$$

Suppose f has a fixed point z_0 with multiplier λ. Then f is not conjugate to a rotation on any neighborhood of z_0, i.e. z_0 is a Cremer point.

Proof. The strategy is to show that any neighborhood of z_0 contains infinitely many periodic points of f, from which it follows that f is not conjugate to a rotation near z_0. See [Milnor, 2006] for details. ■

3.5 Herman rings

We begin with a negative result, to shed some light on why the existence of Herman rings is subtle.

Theorem 3.5.1. *No component of the Fatou set of a polynomial map f is a Herman ring.*

Proof. Suppose, for the sake of contradiction, that U is a Herman ring. Note that $\infty \notin U$, because ∞ is a superattracting point of f (see section 3.2). Since U is disjoint from the basin of infinity, it is contained in some disk $\mathbb{D}_R$. We are assuming that U is conformally equivalent to an annulus, so its complement in $\mathbb{C}$ consists of two components, one of which, K, is compact.

Let $\Gamma \subset U$ be a closed curve, such that the bounded component of the complement of Γ contains K. By the maximum modulus principle, for any $z \in K$ and integer k,

$$|f^{\circ k}(z)| \leq \sup_{w \in \Gamma} |f^{\circ k}(w)| \leq R.$$

Hence the orbit of any point $z \in K$ is bounded. This implies that $z \notin J(f)$, by Proposition 2.5.12 (Iterated images), since z is contained in a neighborhood $N \subset K \cup U$, and the iterated images of such an N are contained in $\mathbb{D}_R$. ■

In the next sections we give a sketch of one construction of a rational map with a Herman ring.

3.5.1 Circle diffeomorphisms and rotation numbers

We consider circle diffeomorphisms $g : S^1 \to S^1$. Since $p : \mathbb{R} \to S^1$, $p(x) = e^{2\pi i x}$ is a covering map, each such diffeomorphism g lifts to a diffeomorphism $G : \mathbb{R} \to \mathbb{R}$ (in many possible ways). We say that g is *orientation preserving* if there is a lift G that is monotone increasing and $G(x+1) = G(x) + 1$ for all $x \in \mathbb{R}$.

Definition 3.5.2. Given an orientation preserving diffeomorphism $g : S^1 \to S^1$, we define the *rotation number* of g to be

$$\left(\lim_{n\to\infty} \frac{G^{\circ n}(x_0)}{n} \right) \bmod 1,$$

where $G : \mathbb{R} \to \mathbb{R}$ is a monotone increasing lift of g with $G(x+1) = x+1$, and $x_0 \in \mathbb{R}$ is arbitrary.

It is not hard to see that the above definition does not depend on the choice of the lift G or the basepoint x_0 (see [Kuehn, 2007]). When g is a rotation $z \mapsto e^{2\pi i\theta} z$, the rotation number is θ, since g lifts to $G(x) = x + \theta$.

We will need the following proposition which allows us to modify a given orientation preserving diffeomorphism by multiplying by a suitable constant to get a orientation preserving diffeomorphism with any desired rotation number.

Proposition 3.5.3. *Let $g : S^1 \to S^1$ be an orientation preserving diffeomorphism. Consider the family*

$$g_\alpha(z) = e^{2\pi i\alpha} g(z)$$

of orientation preserving diffeomorphisms, for $\alpha \in \mathbb{R}$. The function taking α to the rotation number of g_α is continuous, is periodic with period 1, and its range is $[0, 1)$.

The degree to which the rotation number of g can be approximated by rationals has many consequences for various conjugacy properties of g.

Definition 3.5.4. A real number θ is said to be *Diophantine of order n* if there exists $\epsilon > 0$ such that

$$|\theta - p/q| > \epsilon/q^n,$$

for all integers p, q. A number that is Diophantine of order n for some positive integer n is called *Diophantine*.

Remark 3.5.5. By a theorem of Liouville, every non-rational algebraic number θ that satisfies a polynomial with integer coefficients of degree d is Diophantine of order d. Liouville used this result in the first construction of a transcendental number. See [Dunham, 2005].

We will need the following result relating rational approximation to conjugacy to a model map, whose proof we omit.

Theorem 3.5.6. *If $g : S^1 \to S^1$ is a real-analytic, orientation preserving diffeomorphism, with Diophantine rotation number θ, then g is real-analytically conjugate to the rotation $z \mapsto e^{2\pi i\theta} z$.*

3.5.2 Existence

Theorem 3.5.7. *There exists a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ (of degree 3) with a Herman ring.*

Proof. We begin by taking a suitable product p of Blaschke factors taking $\mathbb{D} \to \mathbb{D}$ and $\partial\mathbb{D}$ to $\partial\mathbb{D}$ (see Proposition 2.1.4), together with Möbius transformation similar to Blaschke factors, but taking $\mathbb{D}$ to the complement of $\overline{\mathbb{D}}$, and $\partial\mathbb{D}$ to $\partial\mathbb{D}$. Such a product p preserves $\partial\mathbb{D}$, and can be chosen so that it is "close" to the identity map on $\partial\mathbb{D}$. It then follows that $p|_{\partial\mathbb{D}}$ is an orientation preserving real-analytic diffeomorphism. We can satisfy all these conditions with a map of degree 3.

By Proposition 3.5.3, we can multiply p by some factor $e^{2\pi i\alpha}$ so that the rotation number θ of the restriction of $f := e^{2\pi i\alpha} p$ to $\partial\mathbb{D}$ is Diophantine. By Theorem 3.5.6, on $\partial\mathbb{D}$, there is a real-analytic diffeomorphism h which conjugates f to the rotation $z \mapsto e^{2\pi i\theta} z$. This map h can be extended to a complex analytic map which conjugates f to $z \mapsto e^{2\pi i\theta} z$ on some neighborhood of $\partial\mathbb{D}$. This neighborhood is contained in a rotation domain for f. It cannot be a Siegel disk, since the map f commutes with inversion $z \mapsto \bar{z}$, so if all of $\mathbb{D}$ were in the Fatou set, then the Fatou set would be all of $\widehat{\mathbb{C}}$, contradicting Proposition 2.5.11. ■

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