# States, Link Polynomials, and the Tait Conjectures

*a thesis submitted by*

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*For my grandfather,*
*Louis A. Stanzione*

# Contents

# Introduction

## 1. Motivating Ideas

At its core, this thesis is a study of knots, objects which human beings encounter with extraordinary frequency. We may find them on our person (in our shoelaces and neckties), or around our homes (as entangled telephone cords or Christmas tree lights). History, if not personal experience, teaches us that undoing knots can be a challenging and frustrating task: when Alexander the Great was unable to loose the famous Gordian knot by hand (a feat whose accomplisher, according to an oracle, would come to rule all of Asia), he cheated and cut it open with his sword. Still, a knot's same potential for intricacy serves the needs of scouts, sailors, and mountain climbers very well. Knots even appear in nature: molecular biologists have shown that our DNA strands — the blueprints of our physical being — can be knotted, and some astrophysicists conjecture that the shape of the universe is intimately connected to the geometry of knots. Indeed, the real-world existence of knots lends a certain appeal to their formal study, since many of the problems encountered in knot theory can be formulated in a tangible manner; after all, we can always take out a piece of string and tie a knot. The problems taken up in this thesis are no different ...

Suppose that we take a very long and pliable piece of string and tie a knot in it. Of course, it might be possible that the string is *not* really knotted; for example, if we pull tightly on the ends of a slip knot, the tangle we have made will simply pop out. This situation is not a problem, however, and we shall still refer to the entangled string as a knot. Nonetheless, when constructing these knots, we will refrain from pulling the string taut, thereby leaving the knot relatively loose. Next, we glue the two ends of the string together, creating a knotted, closed loop of string. In fact, this entire process can be done with two, or three, or any number of strings: first, we tie a series of knots involving various combinations of the strings, and then we glue the two ends of each individual string together, forming a collection of knotted, closed loops which are linked (though perhaps not inextricably) together. To avoid being cumbersome, we shall henceforth refer to these knotted, closed loops of strings as *links*, regardless of the number of strings comprising them;[1] furthermore, each piece of string in the link shall be referred to as a *component*.

We can then take our link and lay it out on a sufficiently large table. Because the string we are using is long, pliable, and slack, we can spread the link out until it lies almost flat, apart from those isolated places where one strand, and at *most* one strand, crosses over another. This type of arrangement is called a *link projection*. Three examples of link projections are given in Figure **??**.

[1]A link with only one component is referred to as a *knot*.

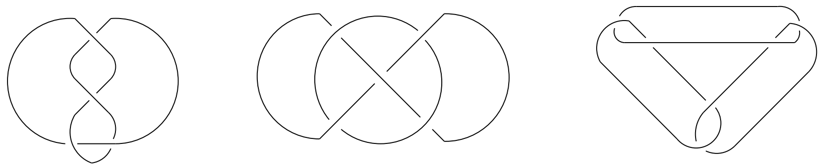

FIGURE 1. Three examples of links projections. The breaks in the diagram indicate where one strand passes under another.

We can now ask the following question: can any of the three links be repositioned on the table so that its new projection possess *fewer* crossings? As it stands now, the first has four crossings, the second has five, and the third has six. And, as it turns out, the answer is no.

There is, in fact, a very special attribute of the first two projections which we can use to conclude that they cannot be simplified: if we imagine traveling along each component in a constant direction (as, say, a monorail would travel along its track), then we would find ourselves, in passing through each crossing, alternately traveling on the over-strand and the under-strand; a projection having this property is said to be *alternating.* The third projection is not alternating: there are two consecutive crossings where a very tiny monorail, riding atop the string, would be on the same level (either over or under) through each. As a result, another method would be needed to verify that this link does not admit a projection with fewer than 6 crossings.

In actuality, alternation is not enough to guarantee that a projection will exhibit the smallest possible number of crossings for that link; one needs two other minor constraints, which we detail in Chapter 2. This result was first formally stated as a conjecture in the late nineteenth century by Peter Guthrie Tait (1831 — 1901), and is now called the First Tait Conjecture. Intuitively, its truth seems plausible: after all, requiring this over-under alternation of the strands adds quite a bit of rigidity to the structure of the link. Surprisingly, it took almost a century to prove.

It is useful to pursue this notion of traveling along each piece of string a bit further — particularly when the crossings in a given projection exhibit the alternating pattern described above. For each component in the link, we choose a direction to travel around it; this is known as *orienting* the link. Clearly, there are only two possible directions in which one can travel around each component, but the choice does not matter. Moreover, we can illustrate the orientation by placing arrowheads on our link along the chosen directions of travel. This is done in Figure **??** for the first two links depicted in Figure **??**.

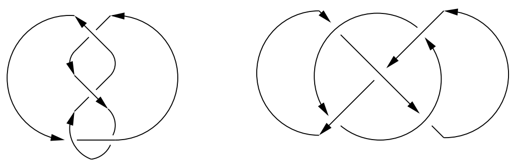

FIGURE 2. Orienting the links.

As a consequence of orienting the link, we can classify every crossing in the various arrangements as safe or unsafe, according to the following convention.[2] Suppose that the under-strand is a service road along which we are driving (in the direction of the arrowheads), and suppose that the over-strand is a highway. If we wish to turn right so as to merge onto the highway, then the flow of traffic along the highway (given by the direction of the arrowheads) should *also* be moving to the right (from our viewpoint on the service road), so as to avoid a crash. If this is the case, the crossing is deemed safe; if not, the crossing is deemed unsafe. The two possible scenarios are illustrated in Figure **??**.

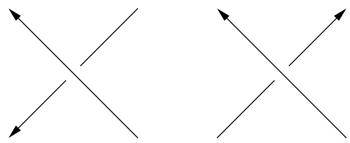

FIGURE 3. A safe crossing (left) and an unsafe crossing (right).

For the first projection in Figure **??**, there are two safe crossings and two unsafe crossings, and for the second, there are two safe crossings and three unsafe crossings. Now, assuming that the arrowheads are securely attached to the string, we can freely rearrange each of the links on the table. Suppose that after such a repositioning is performed, the new projections are *also* alternating. If the projections satisfy the same two additional constraints that are required by the First Tait Conjecture, then the following very subtle observation can be made: the *difference* between the number of safe and unsafe crossings will not change. For example, this difference is 0 for the projection on the left of Figure **??**, and -1 for the projection on the right. This fact was also posited by Tait in the late nineteenth century, and is referred to as the Second Tait Conjecture. Like its sister, the Second Tait Conjecture successfully resisted proof until the 1980s.

To summarize, Tait set down two very elementary conjectures about links possessing alternating projections (provided that the projections satisfy two other very simple requirements). They are as follows:

(1) No other projections can be found with a smaller number of crossings.
(2) If the link is also oriented, then for any two such projections of that link, the difference between the number of safe and unsafe crossings will be the same.

Note that these two facts, taken together, give the following corollary: under the same hypotheses, two projections of an oriented link will have the same number of safe crossings, and the same number of unsafe crossings. Indeed, by the first conjecture, any two such projections will have minimal crossing number; call it $m$. Thus $S + U = m$, where $S$ denotes the number of safe crossings and $U$ denotes the number of unsafe crossings. Moreover, by the second conjecture, $S - U$ is a constant; call it $w$. The system of equations $S + U = m$ and $S - U = w$ has a unique solution, which gives the corollary.

We will prove both the First and Second Tait Conjectures in this thesis. Before we can begin, however, we must formalize some of the ideas presented in these heuristics, and introduce new ones as well.

[2] I was first introduced to this terminology by Curt McMullen; however, it is not standard, and we will, in later chapters, opt for the more standard convention of labeling crossings: safe crossings will be called $+1$ crossings, and unsafe crossings will be called $-1$ crossings.

## 2. Basic Notions in Link Theory

**2.1. Reidemeister Moves and Planar Isotopies.** Much of the experimentation described in the previous section revolved around link projections. Indeed, it should be intuitively clear that any given link can have a multitude of different projections. At first glance, this may seem worrisome, since any conclusions drawn about the link from one of its projections might not hold for any of its other projections. Nonetheless, any two projections of a given link *are* related in the following elegant manner: they can be transformed into each other through a finite sequence of *Reidemeister moves* and *planar isotopies*. This important theorem was first proven in 1932 by Kurt Reidemeister.

A *planar isotopy* is simply a contortion of a link projection; it might twist and bend the projection in severe ways, but it does *not* effect the crossings at all. The *Reidemeister moves*, on the other hand, alter the number *or* arrangement of crossings in projections. There are three general types, and they are depicted in Figure **??**. A Reidemeister I move will either add or remove a kink. A Reidemeister II move will either add or remove two crossings. Finally, a Reidemeister III move will simply rearrange the relative locations of three crossings, without changing the total number. Note that all of these are *local* changes: one can imagine performing planar isotopies or Reidemeister moves in an isolated region of the projection, thereby leaving the projection outside of this region unchanged.

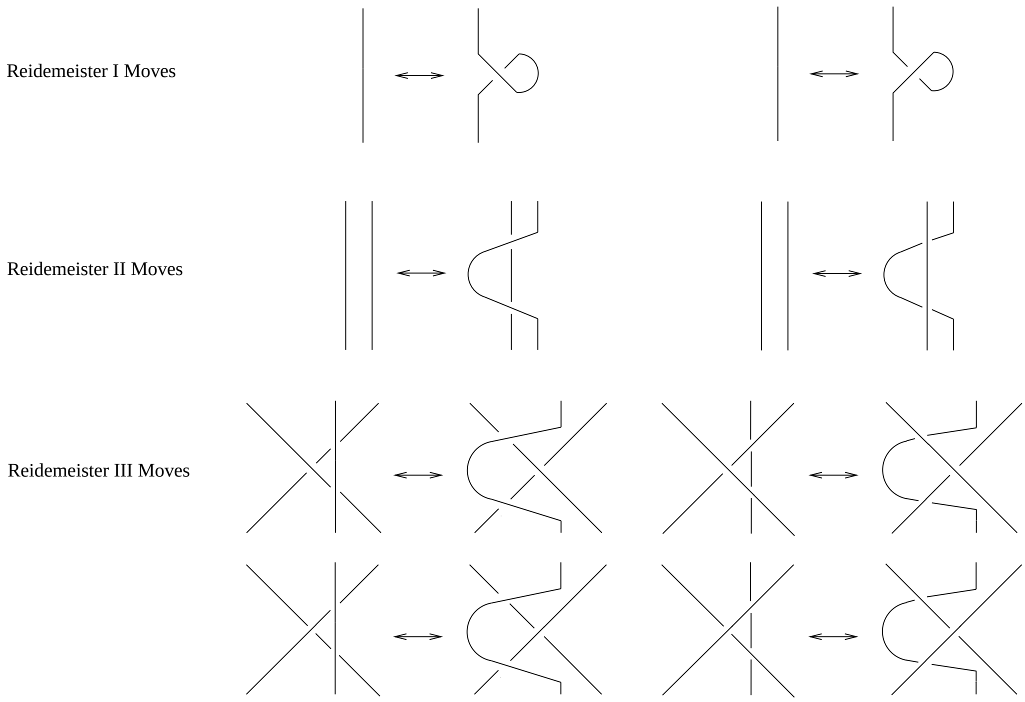


FIGURE 4. The various types of Reidemeister moves.

**2.2. Link Equivalence and Invariants.** Perhaps the central problem of link theory, though certainly not our focus in this thesis, is to decide whether or not two links are the same; in mathematical parlance, we speak of links being *equivalent*. Specifically, much work has been done to detect whether or not a given link is really the *unlink* (i.e., whether or not the link can be completely pulled apart into a collection of disjoint, unknotted loops). At the very least, Reidemeister's theorem gives us a method for approaching the general problem of equivalence: after all, two links are equivalent if and only if a projection of the first can be transformed into a projection of the second (or vice-versa) using a finite sequence of Reidemeister moves and planar isotopies. However, we are really no better off, as the search for such a sequence can be long and tedious; indeed, if the underlying links are *not* equivalent, the search is predestined for failure.

Mathematicians have developed the following, more strategic, approach: they find some characteristic of a link projection which is *unchanged* by the three Reidemeister moves and planar isotopies; such characteristics are called *invariants* of the link. Consequently, if two links are equivalent, then the value of the invariant for each must be the same. Most invariants are numeric (e.g., the number of components in a link). However, some of the most powerful link invariants are actually polynomials, as this thesis will hopefully show.

## 3. Acknowledgements

Before we move on to the substantive mathematics, I would like to express my gratitude to several people for helping me bring this thesis to fruition. First, I would like to thank my advisor, Professor Dylan P. Thurston, for his patience and guidance in overseeing this project. It was a pleasure to work with him.

Peter Green was kind enough to read over many, many revisions. Whatever quality is found in the work is, in large part, due to his extraordinarily constructive feedback.

Thanks must also go to John Boller for his assistance in fleshing out certain proofs, and for also taking the time to comment on several drafts. More importantly, I thank him, along with Susan Milano, for their friendship; my experiences in the Harvard Mathematics Department were all the better for it.

I would like to acknowledge Kathy Paur for vetting the mathematical content, and for suggesting how to improve various aspects of the exposition.

Finally, I would like to pay homage to my proof-readers for mustering the courage to read their first (and perhaps their last?) mathematical paper: Richard Bell, Colin Milburn, Nicole Shadeed, Catherine Sheehan, and Abigail Wild.

CHAPTER 1

# States and Link Polynomials

This chapter introduces the link-theoretic concept of a *state* and uses it to construct three very closely-related link polynomials: the *bracket polynomial*, the *Kauffman polynomial*, and the *Jones polynomial* (in fact, the latter two are identical, up to a change of variables). These three polynomials, in conjunction with states, are key ingredients in the proofs of the Tait Conjectures, as we will see in the next chapter.

## 1. States

States are fundamentally important to the mathematics of this thesis. Their precise definition, however, can be somewhat difficult to motivate, at least from our current standpoint. Nevertheless, they form the cornerstone of much of our work, and their utility will become more apparent as we go on.

States are constructed from link projections using two processes: labeling and cutting; the mechanisms for each are straightforward. Indeed, suppose that $L_\Pi$ is a projection of a link $L$. Then, every crossing in $L_\Pi$ divides a sufficiently small area around that crossing into four regions, which we label either $A$ or $B$, according to the following rule: if the over-strand is rotated slightly counter-clockwise about the crossing point, it should pass over the two $A$ regions; likewise, if the over-strand is rotated slightly clockwise about the crossing point, it should pass over the two $B$ regions. This rule is depicted below in Figure **??**.

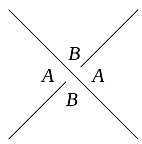


FIGURE 5. The $A$ and $B$ regions around a generic crossing in a link projection.

Informally, we say that a link projection labeled in this manner has been given the $A$-$B$ *labeling*.

EXAMPLE 1.1. The diagram on the left of Figure **??** is a projection of the *trefoil knot*, the simplest non-trivial knot (i.e., link with 1 component). The diagram on the right is the $A$-$B$ labeling of that projection.

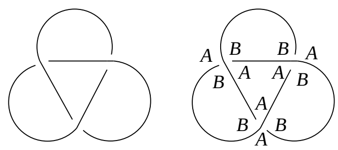


FIGURE 6. A projection of the trefoil knot (left), and its corresponding $A$-$B$ labeling scheme (right).

EXAMPLE 1.2. The diagram on the left of Figure **??** is a projection of the *Hopf link*, the simplest non-trivial link with more than 1 component. The diagram on the right is the $A$-$B$ labeling of that projection.

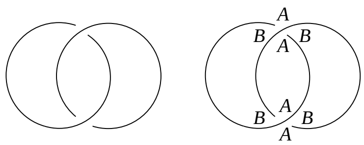


FIGURE 7. A projection of the Hopf link (left), and its corresponding $A$-$B$ labeling scheme (right).

Once a link projection has been given an $A$-$B$ labeling, we can cut open each crossing and splice the strands back together in parallel — thereby eliminating the crossing — in one of two ways: an $A$ *split* removes the barrier between the $A$ regions, while a $B$ *split* removes the barrier between the $B$ regions (see Figure **??**).

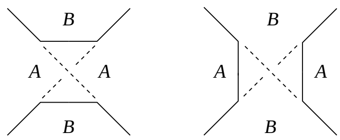


FIGURE 8. An $A$ split (left) and a $B$ split (right). The dashed lines represent the original crossing.

After performing either an $A$ split or a $B$ split at every crossing, we arrive at a projection of the unlink with $n$ components, where $n$ is a positive integer that will, of course, depend on the types of splits that are chosen for each crossing. The projections of the unlink that arise from these processes are what we call states.

DEFINITION 1.3. *Suppose that* $L_\Pi$ *is a projection of a link* $L$ *that has been given the* $A$-$B$ *labeling. Then a* state *of* $L_\Pi$ *is a projection of the unlink on some number of components that is obtained from performing an* $A$ *split or a* $B$ *split at each crossing of* $L_\Pi$. *The collection of all possible states of* $L_\Pi$ *is denoted* $\mathcal{S}_{L_\Pi}$.

EXAMPLE 1.4. Figure **??** shows all of the states of the projection of the trefoil knot depicted in Figure **??**.

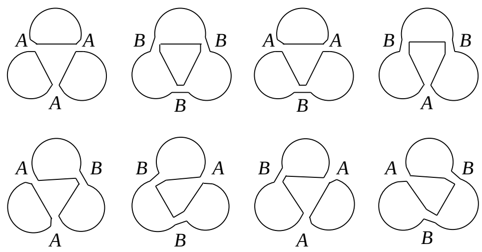

FIGURE 9. The states of the trefoil knot projection depicted in Figure **??**. The type of split performed at each crossing is indicated.

EXAMPLE 1.5. Figure **??** shows all of the states of the projection of the Hopf link depicted in Figure **??**.

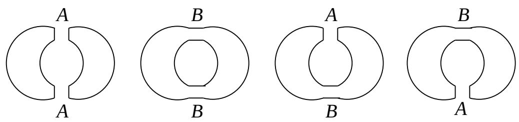

FIGURE 10. The states of the Hopf link projection depicted in Figure **??**. The type of split performed at each crossing is indicated.

At first glance, removing *all* of the crossings in a link projection might not seem like an effective tool for studying it; after all, the crossings determine, to some extent, a link's intrinsic knottedness. However, as we will see in the next section, states *do* succeed in capturing a good deal of information about links.

Before moving on, we digress briefly to give an alternate, set-theoretic formulation of the concept of a state that will be useful during the upcoming proofs of Propositions 1.7 and 1.8. Note that if $L_\Pi$ has $n$ crossings, then there are $2^n$ possible states that can arise from the splitting process. Note further that if we index the $n$ crossings as $\{c_i\}_{i=1}^n$, then each state can be described as a mapping $\{c_1, \ldots, c_n\} \to \{A, B\}$ in the obvious way: namely, $c_i \mapsto A$ if the $i^{th}$ crossing is split in the $A$ manner, and $c_i \mapsto B$ if the $i^{th}$ crossing is split in the $B$ manner.

## 2. The Bracket Polynomial

Now that we know what states *are*, what, exactly, do we do with them? As it turns out, the right answer to this question is to build a link polynomial. The method for doing so is set down in the next definition.

DEFINITION 1.6. *Suppose that $L_\Pi$ is a projection of a link $L$. Let $\mathcal{S}_{L_\Pi}$ denote the collection of all states of $L_\Pi$. For $s \in \mathcal{S}_{L_\Pi}$, let $a(s)$ denote the number of $A$ splits, $b(s)$ denote the number of $B$ splits, and $|s|$ denote the number of components of $s$ (i.e., the number of components in the unlink). Then the* bracket polynomial *of $L_\Pi$, denoted $\langle L_\Pi \rangle[x]$, is a Laurent polynomial in $x$, given by*

$$\langle L_\Pi \rangle[x] := \sum_{s \in \mathcal{S}_{L_\Pi}} x^{a(s)-b(s)}(-x^2 - x^{-2})^{|s|-1}.$$

It is important to note that the bracket polynomial is defined for link *projections*, and not for the links themselves. Therefore, the obvious question to ask is whether or not, for a given link, the bracket polynomial is independent of the projection used to compute it; in other words, is the bracket polynomial, as defined, a link invariant? Unfortunately, it is *not* the case that all of the various projections of a given link will give rise to the same bracket polynomial. To illustrate, consider the two projections $U^1_{\Pi_1}$ and $U^1_{\Pi_2}$ of the unknot depicted in Figure **??**. It is clear that $\langle U^1_{\Pi_1}\rangle[x] = 1$; indeed, $\mathcal{S}_{U^1_{\Pi_1}} = \{U^1_{\Pi_1}\}$ (as there are no crossings to split), and $|U^1_{\Pi_1}| = 1$ (as there is only 1 component). However, we see that $\langle U^1_{\Pi_2}\rangle[x] = x + x^{-1}(-x^2 - x^{-2}) = -x^3$.

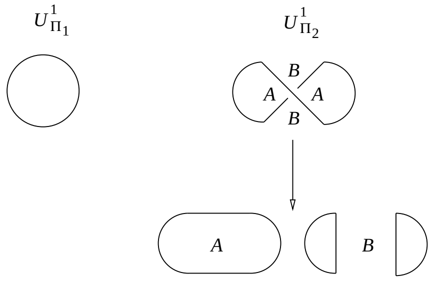


FIGURE 11. Two projections of the unknot, and their respective states (note that the first projection *is* its own state). The type of split performed at each crossing is indicated.

States have not failed us, however, for there *are* several conditions under which two different projections of a link *will* have the same bracket polynomial. First, it is an immediate consequence of Definition 1.6 that if $L_{\Pi_1}$ and $L_{\Pi_2}$ are two projections of a link $L$ that differ by planar isotopy, then $\langle L_{\Pi_1}\rangle[x] = \langle L_{\Pi_2}\rangle[x]$. Furthermore, the bracket polynomial is invariant under Reidemeister moves II and III, as the following two propositions verify.

PROPOSITION 1.7. *Suppose that $L_{\Pi_1}$ and $L_{\Pi_2}$ are two projections of a link $L$ that differ by a Reidemeister II move. Then $\langle L_{\Pi_1}\rangle[x] = \langle L_{\Pi_2}\rangle[x]$.*

PROOF. Assume, without loss of generality, that $L_{\Pi_2}$ has 2 more crossings than $L_{\Pi_1}$. Our goal is to understand how the states of $L_{\Pi_1}$ and $L_{\Pi_2}$ relate to one another. First, focus attention on $\mathcal{S}_{L_{\Pi_2}}$. Assume that $L_{\Pi_2}$ has $n$ crossings. Then, there are $2^n$ possible states of $L_{\Pi_2}$. Index them as $\{s_j\}_{j=1}^{2^n}$, and index the $n$ crossings of $L_{\Pi_2}$ as $\{c_i\}_{i=1}^{n}$, so that $c_1$ and $c_2$ are the two crossings involved in the Reidemeister move (see Figure **??**). With the same slight abuse of notation, let $s_j$ also denote the mapping $s_j : \{c_1, \dots, c_n\} \to \{A, B\}$ which describes the state $s_j$.

Pick any state $s_j \in L_{\Pi_2}$, and consider only the region where the Reidemeister move occurred. In that region, the state will resemble one of the four diagrams (labeled I, II, III, and IV) depicted in Figure **??**, (which diagram it resembles depends, of course, on the values of $s_j(c_1)$ and $s_j(c_2)$). Consequently, there are three other states in $\mathcal{S}_{L_{\Pi_2}}$ (denoted $s_k$, $s_l$, and $s_m$) which correspond to the three other diagrams, and whose function-theoretic descriptions satisfy the following property: $s_j(c_i) = s_k(c_i) = s_l(c_i) = s_m(c_i)$ for all $i \neq 1, 2$. Indeed, we can actually partition $\mathcal{S}_{L_{\Pi_2}}$ into groups of four, so that any four states $s_{j_1}$, $s_{j_2}$, $s_{j_3}$, and $s_{j_4}$ in the same group satisfy the property that $s_{j_1}(c_i) = s_{j_2}(c_i) = s_{j_3}(c_i) = s_{j_4}(c_i)$ for all $i \neq 1, 2$ (i.e., they differ only in the region where the Reidemeister move occurred).

Consider such a group of four states, and let $s_{j_1}$ be the state within the group that corresponds to diagram I, $s_{j_2}$ be the state that corresponds to diagram II, etc.. Note that $s_{j_1}$ and $s_{j_2}$ are planar isotopic, and so $|s_{j_1}| = |s_{j_2}|$. Moreover, $|s_{j_3}| = |s_{j_2}| + 1 = |s_{j_1}| + 1$. Let $s = |s_{j_1}| = |s_{j_2}|$. Then, of the four states in the group, the contribution to $\langle L_{\Pi_2}\rangle[x]$ made by $s_{j_1}$, $s_{j_2}$, and $s_{j_3}$ is:

$$\begin{aligned} x^2(-x^2 - x^{-2})^{s-1} + x^{-2}(-x^2 - x^{-2})^{s-1} + (-x^2 - x^{-2})^s &= \\ (-x^2 - x^{-2})^s \left( \frac{x^2}{-x^2 - x^{-2}} + \frac{x^{-2}}{-x^2 - x^{-2}} + \frac{-x^2 - x^{-2}}{-x^2 - x^{-2}} \right) &= 0 \end{aligned}$$

We conclude from this that the only state in the group which contributes to $\langle L_{\Pi_2}\rangle[x]$ is $s_{j_4}$. But the states $s_j \in \mathcal{S}_{L_{\Pi_2}}$ with $s_j(c_1) = A$ and $s_j(c_2) = B$ (i.e., the states which resemble diagram IV in the region where the Reidemeister move occurred) are in one-to-one correspondence with the states in $\mathcal{S}_{L_{\Pi_1}}$ — indeed, it is obvious from Figure **??** that every diagram IV state in $\mathcal{S}_{L_{\Pi_2}}$ is planar isotopic to a state $s' \in \mathcal{S}_{L_{\Pi_1}}$, and vice-versa. Therefore, these corresponding pairs of states will have the same number of components. Moreover, the quantity $a(s) - b(s)$ for a diagram IV state $s \in \mathcal{S}_{L_{\Pi_2}}$ will equal $a(s') - b(s')$ for the corresponding state $s' \in \mathcal{S}_{L_{\Pi_1}}$, as the single $A$ and single $B$ splitting in diagram IV cancel each other out. From these observations, it immediately follows that $\langle L_{\Pi_1}\rangle[x] = \langle L_{\Pi_2}\rangle[x]$.

The case of the other type of Reidemeister II move is similar. ■

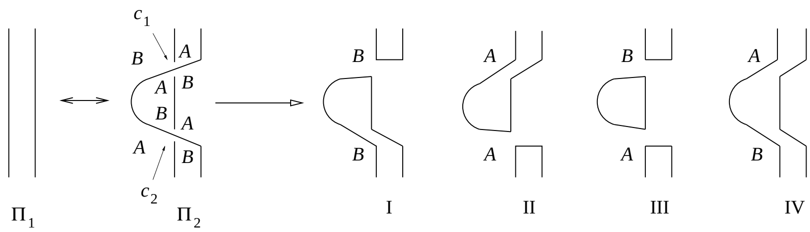


FIGURE 12. Analyzing the effects of a Reidemeister II move on state models. The type of split performed at each crossing is indicated.

PROPOSITION 1.8. *Suppose that $L_{\Pi_1}$ and $L_{\Pi_2}$ are two projections of a link $L$ that differ by a Reidemeister III move. Then $\langle L_{\Pi_1}\rangle[x] = \langle L_{\Pi_2}\rangle[x]$.*

PROOF. As in the proof of Proposition 1.7, our goal is to understand how the states of $L_{\Pi_1}$ and $L_{\Pi_2}$ relate to one another, and we will follow a similar method of justification. First, focus attention on $L_{\Pi_1}$. Assume that $L_{\Pi_1}$ has $n$ crossings. Then, there are $2^n$ possible states of $L_{\Pi_1}$. Index them as $\{s_j\}_{j=1}^{2^n}$, and index the $n$ crossings of $L_{\Pi_1}$ as $\{c_i\}_{i=1}^{n}$, so that $c_1$, $c_2$, and $c_3$ are the three crossings involved in the Reidemeister move (see the top of Figure **??**). With the same slight abuse of notation as in the previous proof, let $s_j$ also denote the mapping $s_j : \{c_1, \ldots, c_n\} \to \{A, B\}$ which describes the state $s_j$.

Pick any state $s_j \in L_{\Pi_1}$, and consider the region where the Reidemeister move occurred. In that region, the state will resemble one of the eight diagrams (labeled I, II, III, …, VIII) depicted in the top half of Figure **??**, (which diagram it resembles depends, of course, on the values of $s_j(c_1)$, $s_j(c_2)$, and $s_j(c_3)$). Consequently, there are seven other states in $\mathcal{S}_{L_{\Pi_1}}$ (denoted $s_k$, $s_l$, $s_m$, $s_n$, $s_o$, $s_p$, $s_q$) which correspond to the seven other diagrams, and whose function-theoretic descriptions satisfy the following property: $s_j(c_i) = s_k(c_i) = s_l(c_i) = s_m(c_i) = s_n(c_i) = s_o(c_i) = s_p(c_i) = s_q(c_i)$ for all $i \neq 1,2,3$. Indeed, we can actually partition $\mathcal{S}_{L_{\Pi_2}}$ into groups of eight, so that any eight states $s_{j_1}, s_{j_2}, \ldots, s_{j_8}$ in the same group satisfy the property that $s_{j_1}(c_i) = s_{j_2}(c_i) = \cdots = s_{j_8}(c_i)$ for all $i \neq 1,2,3$ (i.e., they differ only in the region where the Reidemeister move occurred).

Consider such a group of eight states, and let $s_{j_1}$ be the state within the group that corresponds to diagram I, $s_{j_2}$ be the state that corresponds to diagram II, etc.. Note that $s_{j_6}$ and $s_{j_7}$ are planar isotopic, and so $|s_{j_6}| = |s_{j_7}|$. Moreover, $|s_{j_8}| = |s_{j_7}| + 1 = |s_{j_6}| + 1$. Let $s = |s_{j_6}| = |s_{j_7}|$. Then, of the eight states in the group, the contribution to $\langle L_{\Pi_1}\rangle[x]$ made by $s_{j_6}$, $s_{j_7}$, and $s_{j_8}$ is:

$$\begin{aligned} x(-x^2-x^{-2})^{s-1} + x^{-3}(-x^2-x^{-2})^{s-1} + x^{-1}(-x^2-x^{-2})^{s} &= \\ (-x^2-x^{-2})^{s}\left(\frac{x}{-x^2-x^{-2}} + \frac{x^{-3}}{-x^2-x^{-2}} + \frac{x^{-1}(-x^2-x^{-2})}{-x^2-x^{-2}}\right) &= 0 \end{aligned}$$

Now we turn our attention to $L_{\Pi_2}$, which must also have $n$ crossings. Index them as $\{r_j\}_{j=1}^{2^n}$, and index the $n$ crossings of $L_{\Pi_2}$ as $\{d_i\}_{i=1}^{n}$, so that $d_1$, $d_2$, and $d_3$ are the three crossings involved in the Reidemeister move (see the middle of Figure **??**), and also so that $c_i$ and $d_i$ correspond to the same crossing when $i \neq 1,2,3$. Let $r_j$ also denote the mapping $r_j : \{d_1, \ldots, d_n\} \to \{A, B\}$ which corresponds to the state $r_j$. As before, $\mathcal{S}_{L_{\Pi_2}}$ can be partitioned into groups of eight, so that any eight states $r_{j_1}, r_{j_2}, \ldots, r_{j_8}$ in the same group satisfy the property that $r_{j_1}(d_i) = r_{j_2}(d_i) = \cdots = r_{j_8}(d_i)$ for all $i \neq 1,2,3$. Moreover, a calculation similar to that done for $L_{\Pi_1}$ shows that, of the eight states in any group, those which correspond to diagrams VI', VII', and VIII' together contribute nothing to the bracket polynomial.

To conclude, we return again to $\mathcal{S}_{L_{\Pi_1}}$ and pick any of the groups of eight states; denote its members by $s_{j_1}, s_{j_2}, \ldots, s_{j_8}$, where $s_{j_k}$ corresponds to the $k^{th}$ diagram in the top half of Figure **??**. There is a corresponding group of eight states in $\mathcal{S}_{L_{\Pi_2}}$, whose members are denoted by $r_{j_1}, r_{j_2}, \ldots, r_{j_8}$ (where $r_{j_k}$ corresponds to the $k^{th}$ diagram in the bottom half of Figure **??**), such that $s_{j_1}(c_i) = s_{j_2}(c_i) = \cdots = s_{j_8}(c_i) = r_{j_1}(d_i) = r_{j_2}(d_i) = \cdots = r_{j_8}(d_i)$ for all $i \neq 1,2,3$. As was already noted, $s_{j_6}, s_{j_7}$, and $s_{j_8}$ together do not contribute anything to the bracket polynomial, and neither do $r_{j_6}, r_{j_7}$, and $r_{j_8}$. Moreover, it is clear from Figure **??** that state $s_{j_k}$ is planar isotopic to $r_{j_k}$ for all $1 \leq k \leq 5$, and therefore $|s_{j_k}| = |r_{j_k}|$. In addition, $a(s_{j_k}) - b(s_{j_k}) = a(r_{j_k}) - b(r_{j_k})$ for all $1 \leq k \leq 5$. From these observations, it immediately follows that $\langle L_{\Pi_1}\rangle[x] = \langle L_{\Pi_2}\rangle[x]$.

The cases of the other types of Reidemeister III moves are similar. ■

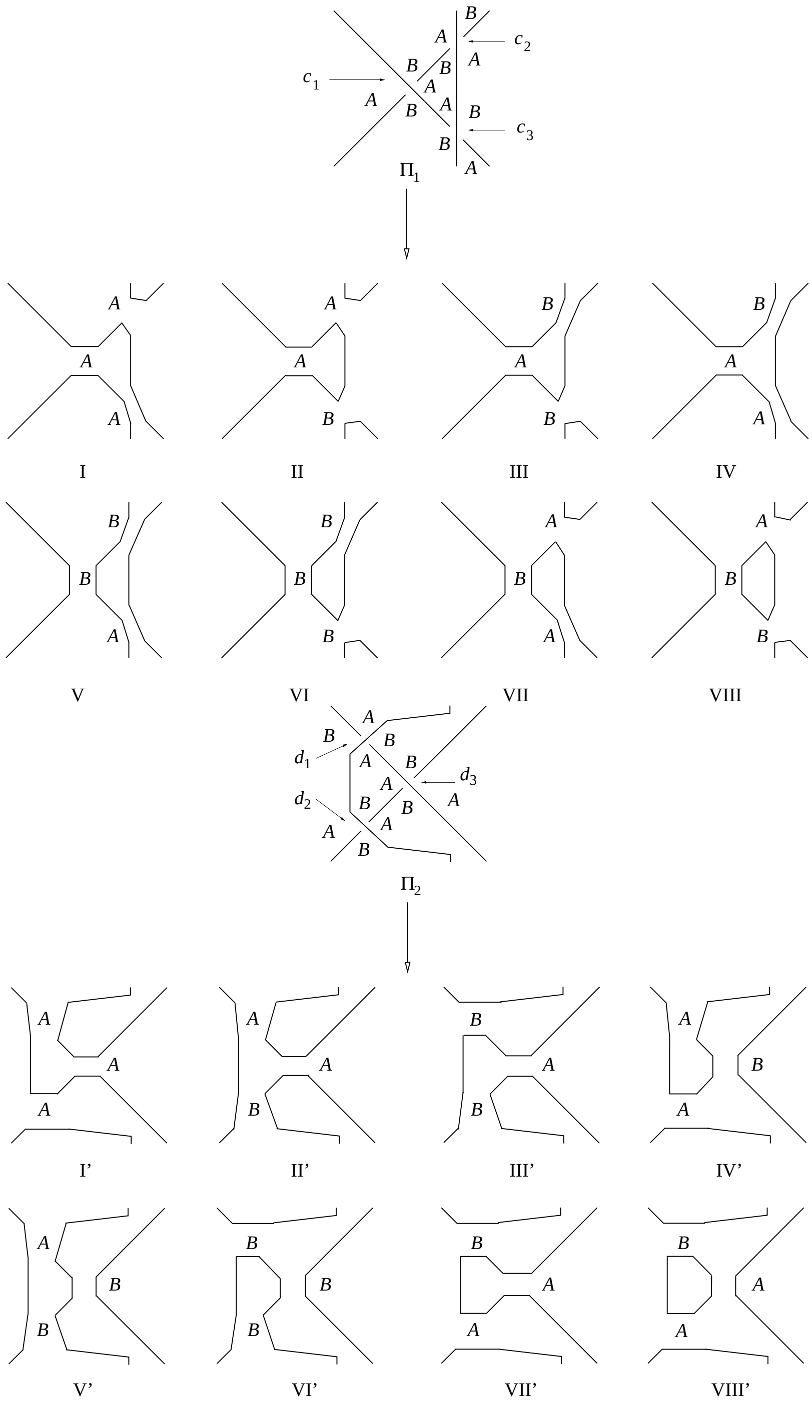


FIGURE 13. Analyzing the effects of a Reidemeister III move on state models. The type of split performed at each crossing is indicated.

The work we have done so far shows that the bracket polynomial is invariant under planar isotopy, as well as Reidemeister moves II and III. However, we noted earlier that the bracket polynomial is *not* a link invariant, and exhibited a situation in which different projections of the same link give rise to different bracket polynomials. Indeed, referring back to Figure **??**, we see that the two given projections of the unknot differ only by a Reidemeister I move. In fact, Reidemeister I moves will always change the bracket polynomial, but in a manner that is easy to quantify algebraically, as the following proposition demonstrates.

PROPOSITION 1.9. *Suppose that $L_{\Pi_1}$ and $L_{\Pi_2}$ are two projections of a link $L$. If $L_{\Pi_1}$ and $L_{\Pi_2}$ differ by the Reidemeister I move that is depicted on the top of Figure* **??***, then $\langle L_{\Pi_2}\rangle[x] = -x^{-3}\langle L_{\Pi_1}\rangle[x]$. If $L_{\Pi_1}$ and $L_{\Pi_2}$ differ by the Reidemeister I move that is depicted on the bottom of Figure* **??***, then $\langle L_{\Pi_2}\rangle[x] = -x^{3}\langle L_{\Pi_1}\rangle[x]$.*

PROOF. Consider the Reidemeister I move that is illustrated on the top of Figure **??**. Note that, given any state of $L_{\Pi_1}$, we can create a state of $L_{\Pi_2}$ by either introducing an artificial $A$ split, or introducing an artificial $B$ split and an additional component (these splits are artificial because there is no crossing in that position in $L_{\Pi_1}$); moreover, all states of $L_{\Pi_2}$ can be regarded as arising in this way. Therefore, if $\langle L_{\Pi_1}\rangle[x] = \sum_{s\in\mathcal{S}_{L_{\Pi_1}}} x^{a(s)-b(s)}(-x^2-x^{-2})^{|s|-1}$, we find that

$$\begin{aligned}
\langle L_{\Pi_2}\rangle[x] &= \sum_{s\in\mathcal{S}_{L_{\Pi_1}}} x^{a(s)+1-b(s)}(-x^2-x^{-2})^{|s|-1} + \sum_{s\in\mathcal{S}_{L_{\Pi_1}}} x^{a(s)-b(s)-1}(-x^2-x^{-2})^{|s|} \\
&= x\left(\sum_{s\in\mathcal{S}_{L_{\Pi_1}}} x^{a(s)-b(s)}(-x^2-x^{-2})^{|s|-1}\right) + \\
&\quad x^{-1}(-x^2-x^{-2})\left(\sum_{s\in\mathcal{S}_{L_{\Pi_1}}} x^{a(s)-b(s)}(-x^2-x^{-2})^{|s|-1}\right) \\
&= -x^{-3}\left(\sum_{s\in\mathcal{S}_{L_{\Pi_1}}} x^{a(s)-b(s)}(-x^2-x^{-2})^{|s|-1}\right) \\
&= -x^{-3}\langle L_{\Pi_1}\rangle[x],
\end{aligned}$$

as desired.

The case of the Reidemeister I move that is illustrated on the bottom of Figure **??** is similar. ■

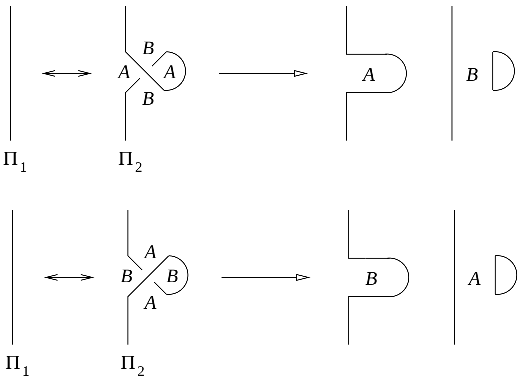


FIGURE 14. Analyzing the effects of the two types of Reidemeister I move on state models. The type of split performed at each crossing is indicated.

Having established the bracket polynomial's elementary properties, let us now compute it for specific links.

EXAMPLE 1.10. For a positive integer $n$, let $U^n$ denote the unlink with $n$ components, and let $U^n_\Pi$ denote the projection of $U^n$ that is depicted in Figure **??**. As there are no crossings in $U^n_\Pi$ to split, $\mathcal{S}_{U^n_\Pi} = \{U^n_\Pi\}$. There are $n$ components, and therefore $\langle U^n_\Pi \rangle[x] = (-x^2 - x^{-2})^{n-1}$.

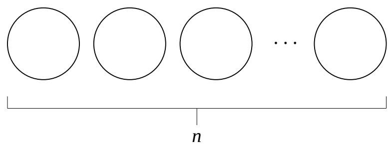


FIGURE 15. A projection of the unlink with $n$ components.

EXAMPLE 1.11. For the trefoil knot $T$, let $T_\Pi$ denote the projection of $T$ that is depicted in Figure **??**, and let $\mathcal{S}_{T_\Pi}$ denote the collection of states of $T_\Pi$ that is depicted in Figure **??**. Then

$$\begin{aligned}\langle T_\Pi \rangle[x] &= x^3(-x^2 - x^{-2})^2 + x^{-3}(-x^2 - x^{-2}) + x(-x^2 - x^{-2}) + x^{-1} + \\ &\quad x(-x^2 - x^{-2}) + x^{-1} + x(-x^2 - x^{-2}) + x^{-1} \\ &= x^7 - x^3 - x^{-5}.\end{aligned}$$

EXAMPLE 1.12. For the Hopf link $H$, let $H_\Pi$ denote the projection of $H$ that is depicted in Figure **??**, and let $\mathcal{S}_{H_\Pi}$ denote the collection of states of $H_\Pi$ that is depicted in Figure **??**. Then

$$\begin{aligned}\langle H_\Pi \rangle[x] &= x^2(-x^2 - x^{-2}) + x^{-2}(-x^2 - x^{-2}) + 1 + 1 \\ &= -x^4 - x^{-4}.\end{aligned}$$

Taken together, Propositions 1.7, 1.8, and 1.9 give us a complete understanding of how the bracket polynomial behaves when the projection used to compute it is altered. And although the polynomial is not an invariant, the extra factors of $-x^{\pm 3}$ induced by Reidemeister I moves are readily dealt with. Still, it would be useful to have a link polynomial which *is* invariant under the full assemblage of Reidemeister moves. It turns out that we *can* modify the bracket polynomial slightly, so as to create a true link invariant; as might be expected, the modification needs only to account for the $-x^{\pm 3}$ terms. Before we can state the modification, however, we need to develop a new tool: the writhe.

## 3. The Writhe

Recall from the introduction that the crossings in a projection of an oriented link fall into one of two categories: safe or unsafe. This terminology, though useful in heuristics, is not standard; instead, link theorists prefer to talk about +1 crossings and −1 crossings, and from now on, we follow their convention. To that end, safe crossings will now be called +1 crossings, and unsafe crossings will be called −1 crossings.

We also described in the introduction a method whereby one can classify a crossing as safe or unsafe (or, using our new terminology, as +1 or −1). A slightly less stylized method is given here: suppose we place our right hand, palm exposed, on top of the projection so that the thumb points in the direction of the over-strand. If our other four fingers point in the direction of the under-strand, then the crossing is assigned the value +1; if not, the crossing is assigned the value −1. The two possible cases are illustrated in Figure **??**.

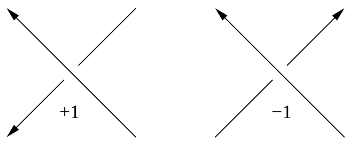


FIGURE 16. A +1 crossing (left) and a −1 crossing (right).

Now we can define the writhe of an oriented link.

DEFINITION 1.13. *Suppose that $\vec{L}_\Pi$ is a projection of an oriented link. Then the* writhe *of $\vec{L}_\Pi$, denoted $w(\vec{L}_\Pi)$, is a sum, taken over all of the crossings, where each term in the sum is either* 1 *or* −1, *depending on whether the crossing is of* +1 *type or* −1 *type, respectively. If there are no crossings in the projection, then the writhe is defined to be 0.*

This formulation might sound familiar — indeed, we encountered it in the introduction as the difference between the number of safe and unsafe crossings.

EXAMPLE 1.14. For the oriented unlink with $n$ components $\vec{U^n}$, let $\vec{U}^n_\Pi$ denote the projection of $\vec{U^n}$ that is depicted on the left of Figure **??**. As there are no crossings, $w(\vec{U}^n_\Pi) = 0$.

EXAMPLE 1.15. For the oriented trefoil knot $\vec{T}$, let $\vec{T}_\Pi$ denote the projection of $\vec{T}$ that is depicted in the middle of Figure **??**. Then $w(\vec{T}_\Pi) = (-1) + (-1) + (-1) = -3$.

EXAMPLE 1.16. For the oriented Hopf link $\vec{H}$, let $\vec{H}_\Pi$ denote the projection of $\vec{H}$ that is depicted on the right of Figure **??**. Then $w(\vec{H}_\Pi) = 1 + 1 = 2$.

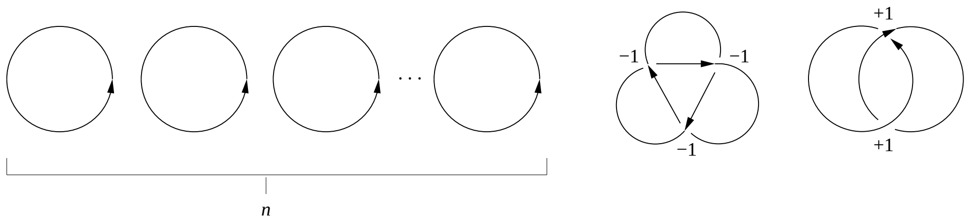


FIGURE 17. A projection of an oriented unlink with $n$ components (left), an oriented trefoil knot (middle), and an oriented Hopf link (right), where the signs of the crossings ($\pm 1$) are indicated.

Because the writhe is computed from a given link projection, we must ask how it will change (if at all) when the projection used to compute it is altered by Reidemeister moves or planar isotopies. Well, the writhe is obviously invariant under planar isotopy. Moreover, like the bracket polynomial, it is invariant under Reidemeister moves II and III, as the following proposition verifies.

PROPOSITION 1.17. *Suppose that* $\vec{L}_{\Pi_1}$ *and* $\vec{L}_{\Pi_2}$ *are two projections of an oriented link* $\vec{L}$ *that differ by a Reidemeister II or III move. Then* $w(\vec{L}_{\Pi_1}) = w(\vec{L}_{\Pi_2})$.

PROOF. Consider the Reidemeister II move that is depicted on the left in Figure **??**. In $L_{\Pi_1}$, there are no crossings to contribute to the writhe. In $L_{\Pi_2}$, there are two crossings, each with opposite signs, and so the contribution is also 0. In fact, this will always be the case for a Reidemeister II move, regardless of the positioning of the strands and their individual orientations.

Next, consider the Reidemeister III move that is depicted on the right in Figure **??**. Note that the effect of the Reidemeister move is simply to relocate two of the three crossings, which does not change the number of $+1$'s and $-1$'s that are present. Hence, the net contribution to the writhe from these three crossings does not change. Again, this will always be the case for Reidemeister III moves, regardless of strand position or orientation. ■

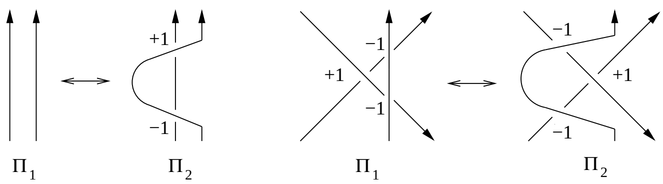


FIGURE 18. A Reidemeister II move (left) and a Reidemeister III move (right). The signs of the crossings ($\pm 1$) are indicated.

The writhe is *not* a link invariant, however, for it is slightly altered when a Reidemeister I move is applied: the effect is simply to increase or decrease it by 1. This is readily seen from Figure **??**, and summarized for convenience in the following proposition.

PROPOSITION 1.18. *Suppose that $\vec{L}_{\Pi_1}$ and $\vec{L}_{\Pi_2}$ are two projections of an oriented link $\vec{L}$. If $\vec{L}_{\Pi_1}$ and $\vec{L}_{\Pi_2}$ differ by the Reidemeister I move that is depicted on the left in Figure* **??**, *then $w(\vec{L}_{\Pi_1}) = w(\vec{L}_{\Pi_2}) - 1$. If $\vec{L}_{\Pi_1}$ and $\vec{L}_{\Pi_2}$ differ by the Reidemeister I move that is depicted on the right in Figure* **??**, *then $w(\vec{L}_{\Pi_1}) = w(\vec{L}_{\Pi_2}) + 1$.* ■

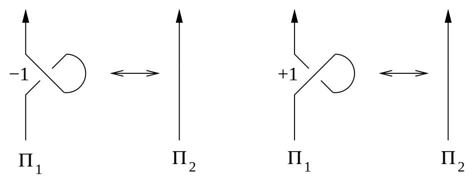


FIGURE 19. Two types of Reidemeister I moves. The signs of the crossings (±1) are indicated.

Let us quickly recall the preliminary results we have established in these last two sections. We have proven that both the writhe and the bracket polynomial are invariant under planar isotopies, Reidemeister II moves, and Reidemeister III moves; nevertheless, neither is a true link invariant, as both undergo small, though predictable, changes under Reidemeister I moves. As we hinted at the end of §1.2, it *is* possible to combine these two objects so that the Reidemeister I effects on the writhe offset the Reidemeister I effects on the bracket polynomial; the Kauffman polynomial — a synthesis of the bracket polynomial and the writhe — achieves just such a cancellation. Moreover, having the Kauffman polynomial at our disposal gives us immediate access to *another* link invariant — one of the most powerful ever discovered: the famous *Jones polynomial.* It can be derived directly from the Kauffman polynomial by a simple change of variables.

## 4. The Kauffman and Jones Polynomials

We go right to the definitions.

DEFINITION 1.19. *Suppose that $\vec{L}_{\Pi}$ is a projection of an oriented link $\vec{L}$, with writhe $w(\vec{L}_{\Pi})$ and bracket polynomial $\langle L_{\Pi}\rangle[x]$. Then the* Kauffman polynomial *of $\vec{L}_{\Pi}$, denoted $K_{\vec{L}_{\Pi}}[x]$, is a Laurent polynomial in $x$, given by*

$$K_{\vec{L}_{\Pi}}[x] := (-x^3)^{-w(\vec{L}_{\Pi})}\langle L_{\Pi}\rangle[x].$$

*Moreover, the* Jones polynomial *of $\vec{L}_{\Pi}$, denoted $V_{\vec{L}_{\Pi}}[t]$, is a Laurent polynomial in $t$, given by*

$$V_{\vec{L}_{\Pi}}[t] := K_{\vec{L}_{\Pi}}[t^{-\frac{1}{4}}].$$

We pause briefly to clarify that the bracket polynomial used in the definition of the Kauffman polynomial is simply the bracket polynomial of the unoriented link $L$. Our first major theorem, proven below, establishes that the Kauffman polynomial (and, hence, the Jones polynomial) is a true link invariant.

THEOREM 1.20. *Suppose that* $\vec{L}_{\Pi_1}$ *and* $\vec{L}_{\Pi_2}$ *are two projections of an oriented link* $\vec{L}$. *Then* $K_{\vec{L}_{\Pi_1}}[x] = K_{\vec{L}_{\Pi_2}}[x]$.

PROOF. Because $\vec{L}_{\Pi_1}$ and $\vec{L}_{\Pi_2}$ are two projections of the same oriented link, there is a finite sequence of Reidemeister moves and planar isotopies that transforms $\vec{L}_{\Pi_1}$ into $\vec{L}_{\Pi_2}$. In other words, there is a finite sequence of projections of $\vec{L}$:

$$\vec{L}_{\Pi_1} = \vec{L}_{\Pi_{1,1}} \to \vec{L}_{\Pi_{1,2}} \to \vec{L}_{\Pi_{1,3}} \to \cdots \to \vec{L}_{\Pi_{1,n-1}} \to \vec{L}_{\Pi_{1,n}} = \vec{L}_{\Pi_2}$$

where $\vec{L}_{\Pi_{1,i+1}}$ is obtained from $\vec{L}_{\Pi_{1,i}}$ by a Reidemeister move or planar isotopy.

If $\vec{L}_{\Pi_{1,i+1}}$ is obtained from $\vec{L}_{\Pi_{1,i}}$ by planar isotopy, then it is an immediate consequence of Definitions 1.6 and 1.13 that $\langle L_{\Pi_{1,i}}\rangle[x] = \langle L_{\Pi_{1,i+1}}\rangle[x]$ and $w(\vec{L}_{\Pi_{1,i}}) = w(\vec{L}_{\Pi_{1,i+1}})$. Therefore, $K_{\vec{L}_{\Pi_{1,i}}}[x] = K_{\vec{L}_{\Pi_{1,i+1}}}[x]$.

If $\vec{L}_{\Pi_{1,i+1}}$ is obtained from $\vec{L}_{\Pi_{1,i}}$ by a Reidemeister II or III move, then by Propositions 1.7 and 1.8, $\langle L_{\Pi_{1,i}}\rangle[x] = \langle L_{\Pi_{1,i+1}}\rangle[x]$. Moreover, by Proposition 1.17, $w(\vec{L}_{\Pi_{1,i}}) = w(\vec{L}_{\Pi_{1,i+1}})$. Again, $K_{\vec{L}_{\Pi_{1,i}}}[x] = K_{\vec{L}_{\Pi_{1,i+1}}}[x]$.

Lastly, suppose that $\vec{L}_{\Pi_{1,i+1}}$ is obtained from $\vec{L}_{\Pi_{1,i}}$ by the Reidemeister I move depicted on the left of Figure **??**; in particular, assume that the twist is being removed. Then $\langle L_{\Pi_{1,i}}\rangle[x] = -x^{-3}\langle L_{\Pi_{1,i+1}}\rangle[x]$ and $w(\vec{L}_{\Pi_{1,i}}) = w(\vec{L}_{\Pi_{1,i+1}}) - 1$ (by Propositions 1.9 and 1.18, respectively). Therefore:

$$\begin{aligned}
K_{\vec{L}_{\Pi_{1,i}}}[x] &= (-x^3)^{-w(\vec{L}_{\Pi_{1,i}})}\langle L_{\Pi_{1,i}}\rangle[x] \\
&= (-x^3)^{-(w(\vec{L}_{\Pi_{1,i+1}})-1)}(-x^{-3})\langle L_{\Pi_{1,i+1}}\rangle[x] \\
&= (-x^3)^{-w(\vec{L}_{\Pi_{1,i+1}})}(-x^{3})(-x^{-3})\langle L_{\Pi_{1,i+1}}\rangle[x] \\
&= (-x^3)^{-w(\vec{L}_{\Pi_{1,i+1}})}\langle L_{\Pi_{1,i+1}}\rangle[x] \\
&= K_{\vec{L}_{\Pi_{1,i+1}}}[x]
\end{aligned}$$

The case in which the twist is being added, and the two cases resulting from the Reidemeister I move depicted on the right of Figure **??**, are similar. ■

COROLLARY 1.21. *Suppose that* $\vec{L}_{\Pi_1}$ *and* $\vec{L}_{\Pi_2}$ *are two projections of an oriented link* $\vec{L}$. *Then* $V_{\vec{L}_{\Pi_1}}[t] = V_{\vec{L}_{\Pi_2}}[t]$. ■

Theorem 1.20 and Corollary 1.21 tell us that all of the projections of an oriented link $\vec{L}$ give rise to the same Kauffman and Jones polynomials. This fact allows us to make the following definitions.

DEFINITION 1.22. *Suppose that* $\vec{L}$ *is an oriented link. Let* $\vec{L}_{\Pi}$ *be* any *projection of* $\vec{L}$, *with Kauffman polynomial* $K_{\vec{L}_{\Pi}}[x]$. *Then the* Kauffman polynomial *of* $\vec{L}$, *denoted* $K_{\vec{L}}[x]$, *is a*

*Laurent polynomial in x, given by*

$$K_{\vec{L}}[x] := K_{\vec{L}_\Pi}[x].$$

*Similarly, the* Jones polynomial *of $\vec{L}$, denoted $V_{\vec{L}}[t]$, is a Laurent polynomial in $t$, given by*

$$V_{\vec{L}}[t] := K_{\vec{L}}[t^{-\frac{1}{4}}].$$

EXAMPLE 1.23. For the oriented unlink with $n$ components $\vec{U^n}$ (Figure **??**), we find, using our work in Examples 1.10 and 1.14, that $K_{\vec{U^n}}[x] = (-x^2 - x^{-2})^{n-1}$, and so $V_{\vec{U^n}}[t] = K_{\vec{U^n}}[t^{-\frac{1}{4}}] = (-t^{-\frac{1}{2}} - t^{\frac{1}{2}})^{n-1}$.

EXAMPLE 1.24. For the oriented trefoil knot $\vec{T}$ (Figure **??**), we find, using our work in Examples 1.11 and 1.15, that $K_{\vec{T}}[x] = (-x^3)^{-(-3)}(x^7 - x^3 - x^{-5}) = -x^{16} + x^{12} + x^4$, and so $V_{\vec{T}}[t] = K_{\vec{T}}[t^{-\frac{1}{4}}] = -t^{-4} + t^{-3} + t^{-1}$.

EXAMPLE 1.25. For the oriented Hopf link $\vec{H}$ (Figure **??**), we find, using our work in Examples 1.12 and 1.16, that $K_{\vec{H}}[x] = (-x^3)^{-2}(-x^4 - x^{-4}) = -x^{-2} - x^{-10}$, and so $V_{\vec{T}}[t] = K_{\vec{T}}[t^{-\frac{1}{4}}] = -t^{\frac{1}{2}} - t^{\frac{5}{2}}$.

It might appear from the organization of our exposition that the Jones polynomial was a theoretical afterthought of the Kauffman (and, therefore, bracket) polynomial. In actuality, it was the Jones polynomial that was discovered first. Surprisingly, Vaughan Jones was working with operator algebras, a branch of mathematics that is very far afield from modern link theory, when he unearthed the polynomial that now bears his name and related it to the study of knots and links. Fortunately, more elementary methods for computing the Jones polynomial, and proving its invariance, were discovered subsequent to Jones' initial breakthrough. Kauffman's scheme (using state diagrams) is one such approach. Indeed, the benefits of using states to analyze links will become even more apparent in the next chapter, when we put these polynomials to work.

CHAPTER 2

# The Tait Conjectures

In this last chapter, we bring our small arsenal of link polynomials, along with the concept of states, to bear on the Tait Conjectures. Recall that these two very famous statements, made by P. G. Tait at the end of the nineteenth century, concern the distinguished class of link projections which exhibit an alternating pattern among their crossings. There is still much to say about such projections; however, we will postpone that discussion for the time being. Indeed, we noted in the introduction that a link projection must satisfy two *additional* conditions, over and above the alternation of crossings, before the Tait Conjectures can be applied to it; these very mild conditions are called *connectedness* and *reduction*. Our task in the next two sections, then, is to come to an understanding of these two very intuitive concepts.

## 1. Connected Projections

To build up to connectedness, we first imagine taking the projections of the trefoil knot and the Hopf link that are given in Figures **??** and **??**, and then filling in all of the breaks that are used to indicate where one strand passes under another. The resulting image is akin to the link casting its shadow down onto the plane (see Figure **??**). Therefore, in all that follows, we will refer to such filled-in projections as *shadows*.

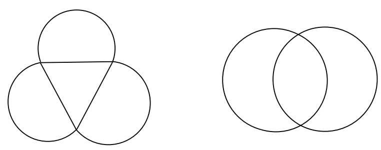

FIGURE 20. The shadows of the trefoil knot projection (left) and Hopf link projection (right) depicted in Figures **??** and **??**.

A projection is said to be connected, then, if we can travel between any two points in the shadow *without* leaving the shadow. We formalize this below.

DEFINITION 2.1. *A projection* $L_{\Pi}$ *of a link* $L$ *is said to be* connected *if, given any two points in the shadow, there exists a path (not necessarily unique) in the shadow that connects the two points.*

EXAMPLE 2.2. The projection of the unlink with 2 components that is depicted on the left of Figure **??** is not connected, as we cannot join the points $R$ and $S$ by a path that lies entirely within its shadow. However, the projection of the same link that is depicted on the right of Figure **??** *is* connected, as such a path (depicted as a heavy line) between $R$ and $S$ now exists; moreover, it is intuitively clear that *any* two points in the shadow can now be connected by paths in the shadow.

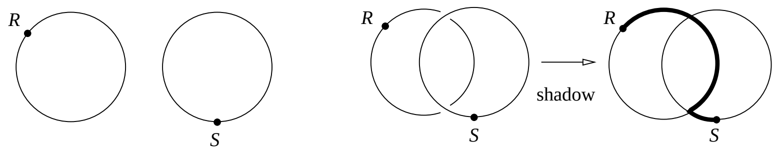


FIGURE 21. Two projections of the unlink with 2 components; the one on the right is connected, while the one on the left is not.

As Figure **??** indicates, it is exceedingly easy to take an unconnected link projection and turn it into a projected one: we simply choose one piece of the link as our base, and then slide each disconnected piece underneath it. Formally, this amounts to performing planar isotopies and Reidemeister II moves.

The usefulness of working with connected projections will become apparent when we begin our proof of the First Tait Conjecture; in the meantime, we proceed directly to the concept of reduction.

## 2. Reduced Projections

Sometimes, a link projection can be unnecessarily complicated. For instance, we can easily imagine taking any link projection and adding several small twists to it using Reidemeister I moves. By assumption, this does nothing to change the underlying link; it just makes the projection messier. The concept of projection reduction uses shadows to identify some of these needless complications.[1]

DEFINITION 2.3. *Suppose that $L_\Pi$ is a projection of a link $L$. Then, the shadow of $L_\Pi$ divides the plane into separate regions. If exactly four distinct regions of the plane meet at every crossing in the shadow, then the projection is said to be* reduced.

EXAMPLE 2.4. The projections of the trefoil knot and Hopf link that are given in Figures **??** and **??**, respectively, are reduced (consult the shadows of these projections in Figure **??**).

The image on the left-hand side of Figure **??** is an unreduced projection of the *granny knot* (built from two trefoils); it has an obvious unnecessary crossing in the middle. Indeed, we see that the region exterior to the shadow meets itself at this crossing. Note that we can draw a path (represented in Figure **??** by a heavy dashed line) from one side of the crossing to another without passing through the shadow; we would not be able to do that if the projection were reduced.

[1]Some complications, however, *are* useful (e.g., adding unnecessary crossings to make a link projection connected).

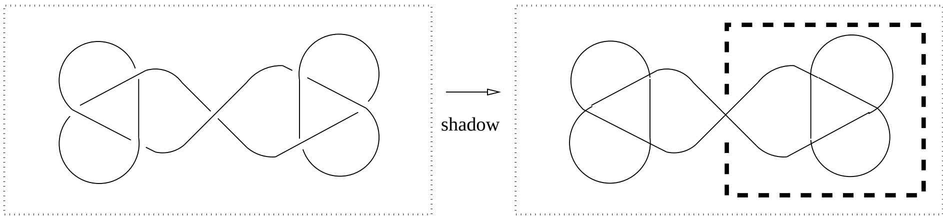


FIGURE 22. A projection of the granny knot which is not reduced. The dotted line represents the plane, and the heavy dashed line represents a path from one side of a crossing to another.

Turning unreduced projections into reduced ones is usually very easy, and the way to do it is often obvious from the projection. For example, the unreduced projection of the granny knot shown in Figure **??** can easily be made into a reduced projection by simply flipping over one of the two trefoils which comprise it.

Since any unreduced projection can easily be made into a reduced projection, insisting that all projections are reduced is not an especially harsh requirement. But why insist at all? One reason is that, without the hypothesis of reduction, the First Tait Conjecture would be false! Figure **??** shows an alternating projection of a link. It is clearly not reduced: taking its shadow, we see that only three regions of the plane meet at the single crossing. Furthermore, this projection does *not* realize the link's minimal crossing number: 0. Mandating reduction in our link projections, then, serves to exclude such annoying pathologies from our analysis.

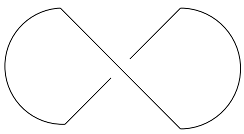

FIGURE 23. An alternating projection which does not realize the minimal crossing number.

Equipped with an understanding of projection connectedness and reduction, we will now quickly review the concept of alternation, and then, at last, state and prove the Tait Conjectures.

## 3. Alternating Links

Recall from the introduction that a link projection is said to be *alternating* if, as we travel around each of its components in a constant direction, we find ourselves, in passing through each crossing, alternately traveling on the over-strand and the under-strand. Note that this definition makes *no* qualifications about whether or not the link is oriented, despite the fact that it talks about traveling around a link. Indeed, a link's orientation has no bearing whatsoever on whether or not it has a projection which is alternating.

Links which possess an alternating projection are called *alternating links*. We should make clear that alternating links *can* have non-alternating projections. For instance, the projection of the trefoil knot shown in Figure **??** is clearly not alternating.

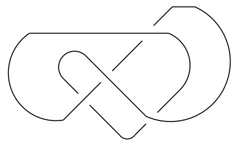

FIGURE 24. A non-alternating projection of the trefoil knot.

We gave two examples of alternating links in the introduction. In addition, the trefoil knot and Hopf link projections that we have been working with (Figures **??** and **??**, respectively) are readily seen to be alternating, whence the trefoil knot and Hopf link are alternating links.

Interestingly enough, if a non-trivial knot can be drawn with 7 or fewer crossings, it *must* be alternating. Indeed, the simplest *non*-alternating knots require projections that have at least 8 crossings.[2] However, it is conjectured that the proportion of knots which are alternating tends to 0 as the minimal crossing number tends to infinity; the data presented in the following table [**?**], which gives the number of alternating and non-alternating knots for a given number of crossings, indicates the plausibility of this claim.[3]

| Number of Crossings | Number of Alternating Knots | Number of Non-Alternating Knots |
|---|---|---|
| 8 | 18 | 3 |
| 9 | 41 | 8 |
| 10 | 123 | 42 |
| 11 | 367 | 185 |
| 12 | 1,288 | 888 |
| 13 | 4,878 | 5,110 |
| 14 | 19,536 | 27,436 |
| 15 | 85,263 | 168,030 |
| 16 | 379,799 | 1,008,906 |

## 4. The Tait Conjectures

Given the ubiquity of alternating knots, it is not surprising that Tait would have given them special attention in his research.[4] In 1898, he published a series of papers containing his now-famous conjectures regarding alternating knots; in this thesis, we concern ourselves only with the first two.[5] In fact, all three remained unproven until the 1980s, subsequent to the discovery of the Jones polynomial, some of whose properties are used in their proofs. As it turns out, the conjectures are true for alternating links (i.e., we need not restrict ourselves to one component), and so we will state and prove them with that generality.

[2]The first proof that a non-alternating knot actually *existed* was not found until 1930 [**?**].

[3]It *is* known, however, that the proportion of alternating *links* tends to 0 very fast as the minimal crossing number increases.

[4]Tait's work in knot theory was motivated in large part by the belief, prevalent in the nineteenth century, that atoms were actually knotted rings [**?**]. Therefore, a complete classification of the natural elements mandated a complete classification of knots.

[5]We will, however, describe the third conjecture in §2.8.

Theorem 2.5 (First Tait Conjecture). *Suppose that $L_{\Pi_1}$ and $L_{\Pi_2}$ are two connected projections of a link L. Suppose further that $L_{\Pi_1}$ is an alternating and reduced projection. If $L_{\Pi_1}$ has $n_1$ crossings and $L_{\Pi_2}$ has $n_2$ crossings, then $n_1 \leq n_2$.*

(Note that if $L$ is a knot, then there is no need to require that the projection be connected.)

Theorem 2.6 (Second Tait Conjecture). *Suppose that $\vec{L}_{\Pi_1}$ and $\vec{L}_{\Pi_2}$ are two alternating, connected, reduced projections of an oriented link $\vec{L}$. Let $w(\vec{L}_{\Pi_1})$ and $w(\vec{L}_{\Pi_2})$ denote the writhes of $\vec{L}_{\Pi_1}$ and $\vec{L}_{\Pi_2}$, respectively. Then $w(\vec{L}_{\Pi_1}) = w(\vec{L}_{\Pi_2})$.*

The task at hand, then, is to give complete proofs of these theorems.

## 5. Proof of the First Tait Conjecture

The proof will be broken down into several steps, and we will (for the most part) set our compass by Lickorish's excellent text [**?**]. First, we will take a closer look at how link projections partition the planes in which they lie; this exercise lays the groundwork for much that follows. Then, we will use states to analyze the geometry of alternating links — a geometry that can be articulated quite nicely through link polynomials; indeed, this is the fundamental insight which enables us to prove Theorem 2.5.

### 5.1. Step 1: Partitioning the Plane.

Reconsider the shadows of the trefoil knot and Hopf link projections that are depicted in Figure **??**. Place each shadow in its own copy of the plane. In so doing, it is transparent that these shadows will partition the plane into disjoint regions: 5 in the case of the trefoil knot, and 4 in the case of the Hopf link (we must include the unbounded, infinite region which is exterior to the shadow when we count). Likewise, the shadow of the granny knot (depicted in Figure **??**) divides the plane into 9 regions. Notice, though, that if we subtract 2 from each of 5, 3, and 9, we recover the number of crossings in the projections of the trefoil knot, Hopf link, and granny knot, respectively.

At first glance, we might guess that the shadow of a link projection with $m$ crossings will divide the ambient plane into $m+2$ regions; however, this is incorrect. To see why, consider the standard projection $U^n_\Pi$ of the unlink with $n$ components (depicted in Figure **??**). This projection (which, of course, *is* its own shadow) has 0 crossings. However, it divides the plane into only $n+1$ regions. A moment's thought reveals what the problem is: *the projection $U^n_\Pi$ is not connected!* Indeed, if we transform the standard projection of the unlink into a connected projection (as is done in Figure **??** when $n = 2$), then the formula *will* hold. This result will come in handy later on, and so we record (and prove) it below.

Proposition 2.7. *Let $L_\Pi$ be a connected projection of a link L that has n crossings. Then the shadow of $L_\Pi$ partitions the plane into $n+2$ distinct regions.*

Proof. Imagine taking a piece of tracing paper and placing it on top of $L_\Pi$. Starting over a point which is *not* a crossing in the original projection, we begin tracing out the link, always taking care to stay on the shadow of the same component (i.e., we never make any sharp turns at a crossing). If this component never crosses itself, we will eventually return to where we started, and the curve we have traced out will have an inside and an outside.[6]

Suppose, instead, that the component we are currently on crosses itself $m$ times in the original projection; then, as we trace it out on our paper, we will necessarily have to cross our path. Every time we do so, we enclose 1 region. At some point, we will have reproduced all $m$ crossings on our tracing paper, having enclosed $m$ regions in the process. As we close up the component, however, by arriving back at our starting point, we enclose 1 more, giving a total of $m + 1$ enclosed regions. However, we must not forget about the infinite region! Including it in our tally brings the total number of planar regions to $m + 2$. Indeed, notice that if $m = 0$, then the formula agrees with the conclusion of the previous paragraph — namely, that a curve with no crossings divides the plane into two regions.

If the link has only one component, we are done. If not, we find another component whose shadow crosses the shadow of the component we have just drawn; we can do this because the projection is connected. Next, we choose a starting point, *and* a direction in which to trace, so that the first *new* crossing we make on our tracing paper will occur when we cross the shadow that has *already* been traced out. Thus, we have added 1 crossing, bringing the total number of crossings on our tracing paper to $m + 1$, but we have *not* enclosed any new regions. However, when we add a *second* new crossing,[7] we *must* also enclose a new region. The general pattern is as follows: as we continue tracing out the second component's shadow, adding an $n^{th}$ crossing will enclose an $n - 1^{th}$ new region. The lagging effect, however, is compensated for when we come back to our starting point on the second component's shadow: that will *not* add a new crossing, but it *will* enclose a new region. Among the traced-out shadows of two of the components, then, we see $m+n$ crossings, and $m+n+2$ regions. Note that we do not double-count the infinite region; although the shape of its boundary might have changed, we already accounted for it in the previous paragraph.

It is now easy to see how the result follows: we simply induct on the number of components in the link, using the algorithm described in the previous paragraph. Each new tracing will add $k$ crossings and $k$ enclosed regions. ■

We can make another important observation about these partitioned regions. To see what that is, we superimpose the $A$-$B$ labels from the original projections on top of their shadows (see the two left-most diagrams in Figure **??**). Notice that every region contains only labels of the same type. This consistency, in turn, enables us to label each *region* $A$ or $B$, according to the following (predictable) rule: namely, the region is labeled $A$ if there are $A$ labels inside of it, and labeled $B$ if there are $B$ labels inside of it (see the two right-most diagrams in Figure **??**). Of course, we must not forget to also label the region exterior to the shadow.

Although we have exhibited the $A$-$B$ labeling consistency only in these two particular cases, the fact does generalize to all *connected*, alternating link projections. Indeed, it is easy to construct examples of alternating link projections which fail to have labeling consistency because they are disconnected. For instance, the left-hand side of Figure **??** depicts an alternating link with four components. However, if we take the projection's shadow, superimpose

[6]A thoroughly rigorous proof of this statement would require a weaker version of a famous topological result known as the *Jordan Curve Theorem*.

[7]There must be another crossing, by the Jordan Curve Theorem.

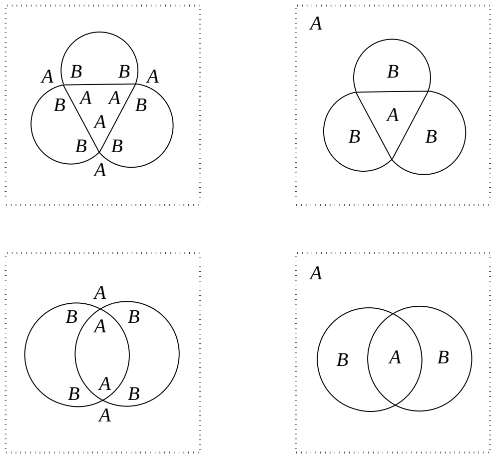

FIGURE 25. Labeling the regions created by shadows. The dotted lines represent separate copies of the plane.

the $A$-$B$ labeling, and use it, in turn, to label each of the regions, we will be unable to choose a label for the region exterior to the shadow.

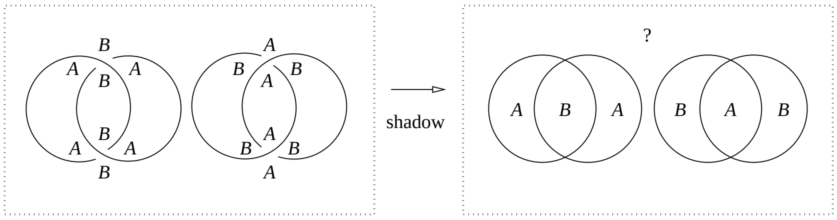


FIGURE 26. A link with 4 components which does not have a consistent $A$-$B$ labeling. The dotted line represents the plane.

Nonetheless, if a link projection is alternating and connected, then the corresponding labeling of the plane *will* be consistent, as the following lemma verifies.

LEMMA 2.8. *Suppose that $L_\Pi$ is an alternating and connected projection of a link $L$. The the shadow of $L_\Pi$ partitions the plane into regions that inherit a consistent labeling from the $A$-$B$ labeling of $L_\Pi$.*

PROOF. Consider first the regions that have finite area. Before the breaks in the projection were filled in, the boundary of such a region contained a certain number of crossings (see the top of Figure **??**). We now pretend that we are standing atop one of the crossings — and, without loss of generality, that we are on the over-strand. We then walk along this strand in the direction which keeps us on the boundary (anti-clockwise, in the case of Figure **??**). We will eventually come to another crossing; moreover, since we were on the over-strand at the previous crossing, we must, by hypothesis, be on the under-strand at the present crossing. To continue, we hop up to the over-strand (which might put us on a new component — but no matter), and walk in the same general direction along the boundary of the region until we come to the next crossing. When we arrive there, we will again be on the under-strand. We hop up to the over-strand, and move on.

This heuristic tells us that the strands comprising the boundary are layered. More precisely, if we ignore the rest of the projection, focusing only on the immediate vicinity of our enclosed region (as in Figure **??**), we will see $m$ crossings and $m$ strands. Moreover, there will be a consistent direction of travel (anti-clockwise, in the case of Figure **??**) around the region so that the strand we are currently on will always dive *under* the next strand (i.e., the strands are layered). Once we have this layered structure, we establish labeling consistency by inducting on the number of crossings, using the template on the bottom of Figure **??** (which captures the layering) to attach each additional crossing.[8]

To conclude, we simply note that the same exact argument works for the unbounded region of the plane that lies exterior to the shadow. ■

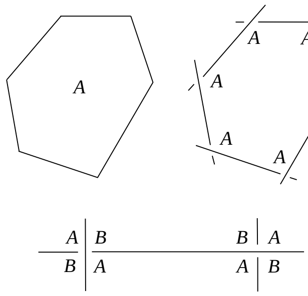


FIGURE 27. Proving the consistency of the $A$-$B$ labeling in a given region of the plane.

**5.2. Step 2: The Adequacy of Alternating Projections.** The proof of Theorem 2.5 continues with an investigation of the additional geometric structure that alternation imparts on states — a structure we will soon exploit through our link polynomials. To these ends, the concept of *adequacy* proves very useful.

DEFINITION 2.9. *Suppose that $L_\Pi$ is a projection of a link $L$, and let $\mathcal{S}_{L_\Pi}$ denote the collection of all possible states of $L_\Pi$. Let $s_A \in \mathcal{S}_{L_\Pi}$ denote the state of $L_\Pi$ in which all crossings are $A$ split, and let $s_B \in \mathcal{S}_{L_\Pi}$ denote the state of $L_\Pi$ in which all crossings are $B$ split. If, for all states $s \in \mathcal{S}_{L_\Pi}$ that have exactly one $B$ split, we have that $|s_A| > |s|$, then $L_\Pi$ is said to be* plus-adequate. *If, for all states $s \in \mathcal{S}_{L_\Pi}$ that have exactly one $A$ split, we have that $|s_B| > |s|$, then $L_\Pi$ is said to be* minus-adequate. *If $L_\Pi$ is both plus-adequate and minus-adequate, $L_\Pi$ is said to be* adequate.

EXAMPLE 2.10. It is easy to see by inspecting the states depicted in Figures **??** and **??** that the projections of the trefoil knot and Hopf link given in Figures **??** and **??** are both adequate.

In order to check whether or not a link projection is plus-adequate, we first draw the state in which all crossings undergo $A$ splits. The result, as in any state, is an unlink with some number of components. In order for the projection to be plus-adequate, a switch from an $A$ split to a $B$ split at any one crossing should decrease the total number of components in the unlink. What if the switch increased the number? The only conceivable way in which that could happen is if a component met itself at the site of a former crossing (see the left-hand side of Figure **??**). Indeed, if this were the case, then a switch from an $A$ to a $B$ split at that site would transform one component into two, with one component nested inside the other. Thus, if no component in $s_A$ meets itself at the site of a former crossing, then the projection *will* be plus-adequate. A similar line of reasoning shows that a projection will fail

[8]Note that in Figure **??** we ignored any possible curvature in the link projection. This is a standard and valid method in topology to aid in both visualization and proof.

to be minus-adequate if and only if some component of $s_B$ meets itself at the site of a former crossing (see the right-hand side of Figure **??**).

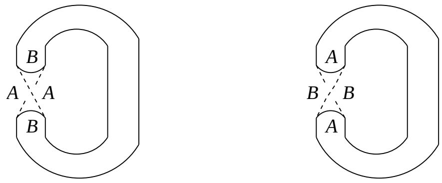


FIGURE 28. A state which fails to be plus-adequate (left) and a state which fails to be minus-adequate (right). The dashed lines represent the original crossings.

As it turns out, every reduced, alternating projection of an alternating link is adequate, as we will now prove.

LEMMA 2.11. *Suppose that $L_\Pi$ is an alternating, connected, and reduced projection of a link $L$. Then $L_\Pi$ is adequate.*

PROOF. Clearly, the shadow of $L_\Pi$ partitions the plane into several bounded (and one unbounded) regions. Because the projection is alternating and connected, Lemma 2.8 says that we can use the $A$-$B$ labeling of the projection to give an $A$ or $B$ label to each of those regions.

We want to understand what happens when we perform an $A$ split at each crossing. To facilitate this, imagine that the $A$ regions are filled with water. Note that some of these bodies of water might have infinite area (refer to Figure **??**); this is not a problem. Further imagine that each of the crossings in the shadow are dams. Performing an $A$ split is analogous to allowing the water to break through the dams, thereby transforming the $B$ regions into little islands.[9] Likewise, if we imagine that the $B$ regions are filled with water, then performing $B$ splits will transform the $A$ regions into islands.

It is important to note that the number of connected components in $s_A$ corresponds exactly to the number of shorelines created when the water in the $A$ regions bursts through the dams. Similarly, the number of connected components in $s_B$ corresponds exactly to the number of shorelines created when the water in the $B$ regions bursts through the dams.

Suppose that we wish to switch one of the splits in $s_A$ from $A$ to $B$. From an aerial point-of-view, this procedure is analogous to constructing a very wide land bridge over one of the dams which connects two of the islands. We should think of this land bridge as extending the shoreline.

What could cause the number of shorelines to increase when we add this bridge? Well, if it turned out that we were building a bridge from an island to itself (as would be the case, for instance, if one of the islands resembled the diagram on the left-hand side of Figure **??**), then we would create another shoreline (in Figure **??**, there would be a shoreline on the inner part of the island, and another on the outer part). Indeed, a moment's thought reveals that building a bridge from an island to itself is the only way to increase the number of shorelines (i.e., the number of connected components in the state) — the only other cases to consider

[9]This analogy is borrowed from Adams' book [**?**].

(connecting two islands of finite area, and connecting an island of finite area with an island of infinite area) clearly cannot work.

It is, however, impossible to connect an island to itself; this is where we use the assumption of reduction. Notice that in $s_A$, every island corresponds to a $B$ region of the plane (and this region may or may not be unbounded — it does not matter). Building a bridge from an island to itself would require that the island meet itself at one of the dams. However, the assumption of having a reduced diagram precludes this from happening. We have therefore verified plus-adequacy.

A similar argument works to check minus-adequacy, and the result then follows. ■

**5.3. Step 3: The Bracket Polynomial – A Reprise.** The adequacy of an alternating, connected, reduced projection is a very strong condition. As we will now show, it gives bounds on the minimum and maximum degrees of the projection's bracket polynomial. These bounds are useful precisely because they involve the number of crossings in the projection.

LEMMA 2.12. *Suppose that $L_\Pi$ is a projection of a link $L$ with $n$ crossings. Let $\langle L_\Pi\rangle$ denote the bracket polynomial of $L_\Pi$. Let $\max\langle L_\Pi\rangle[x]$ and $\min\langle L_\Pi\rangle[x]$ denote, respectively, the maximum and minimum degrees of $\langle L_\Pi\rangle$. Let $s_A \in \mathcal{S}_{L_\Pi}$ denote the state of $L_\Pi$ in which all crossings are $A$ split, and let $s_B \in \mathcal{S}_{L_\Pi}$ denote the state of $L_\Pi$ in which all crossings are $B$ split. Then $\max\langle L_\Pi\rangle[x] \leq 2(|s_A|-1)+n$, with equality if $L_\Pi$ is plus-adequate, and $\min\langle L_\Pi\rangle[x] \geq -2(|s_B|-1)-n$, with equality if $L_\Pi$ is minus-adequate.*

PROOF. We will prove the statement regarding the maximum degree of $\langle L_\Pi\rangle$; the proof for the minimum degree of $\langle L_\Pi\rangle$ is completely analogous. Begin by noting that the contribution of $s_A$ to $\langle L_\Pi\rangle$ is $x^n(-x^2-x^{-2})^{|s_A|-1}$. Clearly, the maximum power of $x$ in this expression is $2(|s_A|-1)+n$. We claim that no other $s \in \mathcal{S}_{L_\Pi}$ contributes a term to $\langle L_\Pi\rangle$ in which $x$ is raised to a power greater than $2(|s_A|-1)+n$.

Note that any state $s \in \mathcal{S}_{L_\Pi}$ can be constructed from $s_A$ by switching certain $A$ splits to $B$ splits, one at a time; put differently, there is a sequence $s_A = s_0, s_1, s_2, \ldots, s_j = s$, where $s_{i+1}$ is obtained from $s_i$ by a single switch of an $A$ split to a $B$ split (the index $i$ counts the number of $B$ splits). Because the states differ in only one choice of splitting, it must be the case that $|s_{i+1}| = |s_i| \pm 1$. Thus, if $s_i$ contributes $x^{a(s_i)-b(s_i)}(-x^2-x^{-2})^{|s_i|-1}$ to $\langle L_\Pi\rangle$, then $s_{i+1}$ contributes:

$$x^{a(s_i)-1-(b(s_i)+1)}(-x^2-x^{-2})^{(|s_i|\pm1)-1} = x^{a(s_i)-b(s_i)-2}(-x^2-x^{-2})^{(|s_i|\pm1)-1}$$

The highest power of $x$ in the $s_i$ contribution is $a(s_i)-b(s_i)+2|s_i|-2$. If $|s_{i+1}| = |s_i|+1$, then the highest power of $x$ in the $s_{i+1}$ contribution is $a(s_i)-b(s)+2|s_i|-2$. If $|s_{i+1}| = |s_i|-1$, then the highest power of $x$ in the $s_{i+1}$ contribution is $a(s_i)-b(s_i)+2|s_i|-6$. We observe from these calculations that the highest power of $x$ in $x^{a(s_{i+1})-b(s_{i+1})}(-x^2-x^{-2})^{|s_{i+1}|-1}$ is either the same, or 4 less than, the highest power of $x$ in $x^{a(s_i)-b(s_i)}(-x^2-x^{-2})^{|s_i|-1}$.

It follows immediately that the highest power of $x$ in $x^{a(s)-b(s)}(-x^2-x^{-2})^{|s|-1}$ will be the same, or strictly less than, $2(|s_A|-1)+n$. As $s$ can be any state, it follows that $\max\langle L_\Pi\rangle[x] \leq 2(|s_A|-1)+n$. Indeed, if $s_A$ is plus-adequate, then when we go from $s_A = s_0$ to $s_1$, it must be that $|s_1| = |s_A|-1$, in which case the highest power of $x$ in $x^{a(s_1)-b(s_1)}(-x^2-x^{-2})^{|s_1|-1}$

must be strictly *less* than $2(|s_A|-1)+n$ for all $i \geq 1$. Thus, for all $s \neq s_A \in \mathcal{S}_{L_\Pi}$, the highest power of $x$ in $x^{a(s)-b(s)}(-x^2-x^{-2})^{|s|-1}$ will be strictly less than $2(|s_A|-1)+n$, and so it follows directly that $\max\langle L_\Pi\rangle[x] = 2(|s_A|-1)+n$ ■

Note that the connectedness of the projection $L_\Pi$ was not required in the above proof. However, this assumption is required again in the following lemma, which establishes an even stronger relationship between the quantities $|s_A|$, $|s_B|$, and the number of crossings in $L_\Pi$. Here, however, we can drop the requirement of reduction.

LEMMA 2.13. *Suppose that $L_\Pi$ is a connected projection of a link $L$ with $n$ crossings. Let $s_A \in \mathcal{S}_{L_\Pi}$ denote the state of $L_\Pi$ in which all crossings are $A$ split, and let $s_B \in \mathcal{S}_{L_\Pi}$ denote the state of $L_\Pi$ in which all crossings are $B$ split. Then $|s_A| + |s_B| \leq n+2$, with equality if $L_\Pi$ is alternating.*

PROOF. We prove the inequality by induction on $n$. Suppose that $L_\Pi$ projection has 0 crossings. Clearly any such (connected) projection is planar isotopic to the standard projection of the unknot (i.e., $L = U^1_\Pi$ in Figure **??**). Here, $s_A = s_B$, and $|s_A| = |s_B| = 1$, and thus $1+1 \leq 0+2$, as required. (Again, if we did not require that the projection was connected, then the projection of the unlink with 2 or more components, as given in Figure **??**, would invalidate our claim.)

Now assume that the result is true for all $m \in \mathbb{N}$ with $0 \leq m \leq n-1$. Let $L_\Pi$ have $n$ crossings. Select any crossing of $L_\Pi$. Clearly, performing either type of split at this crossing will produce a projection, denoted $L'_{\Pi'}$, of a new link $L'$; of course $L'_{\Pi'}$ depends on the type of split performed, but it will most certainly have $n-1$ crossings. It follows from the connectedness of $L_\Pi$ that for some choice of splitting (either $A$ or $B$), $L'_{\Pi'}$ will also be connected. Indeed, without loss of generality, suppose that an $A$ split achieves a connected $L'_{\Pi'}$. Thus, since $L'_{\Pi'}$ is connected and has $n-1$ crossings, it satisfies the inductive hypothesis. If we let $s'_A$ and $s'_B$ denote, respectively, the states of $L'_{\Pi'}$ in which all of the $n-1$ crossings are $A$ split or $B$ split, then it is immediate that $s_A = s'_A$ and $|s_B| = |s'_B| \pm 1$. Using the inductive hypothesis, we have that $|s_A| + |s_B| = |s'_A| + |s'_B| \pm 1 \leq (n-1)+2 \pm 1 \leq n+1 \pm 1 \leq n+2$, as desired.

Now assume that $L_\Pi$ is alternating. Then, everything discussed in the first three paragraphs of the proof of Lemma 2.11 is valid, as reduction of the projection had not yet been invoked. Consider, then, the shadow of $L_\Pi$, and the labeling it induces on the plane. Using the geographical analogy introduced in the proof of Lemma 2.11, we recall that $|s_A|$ counts the number of shorelines created when the $A$ regions, filled with water, were allowed to burst through the dams. Moreover, every shoreline also corresponds to a unique island, and vice versa. If an island had more than one shoreline, then there would be a lake somewhere in its interior, which violates the fact that the $A$ regions were opened to allow the bodies of water to become contiguous. Since the islands, in this case, are the $B$ regions, $|s_A|$ equals the number of $B$ regions. Similarly, $|s_B|$ equals the number of $A$ regions, and so $|s_A|+|s_B|$ equals the total number of regions into which the shadow of $L_\Pi$ divides the plane. By Proposition 2.7, this number is equal to $n+2$, where $n$ denotes the number of crossings in $L_\Pi$ — hence the result. ■

**5.4. Step 4: The Kauffman and Jones Polynomials — A Reprise.** Only one more auxiliary lemma is needed before we can complete our proof of Theorem 2.5; it calls upon the Kauffman and Jones polynomials.

Lemma 2.14. *Suppose that $\vec{L}_\Pi$ is a connected projection of an oriented link $\vec{L}$. Suppose further that $\vec{L}_\Pi$ has $n$ crossings. Let $V_{\vec{L}}[t]$ denote the Jones polynomial of $\vec{L}$, computed using $\vec{L}_\Pi$. Let* $\max V_{\vec{L}}[t]$ *and* $\min V_{\vec{L}}[t]$ *denote, respectively, the maximum and minimum degrees of $V_{\vec{L}}[t]$, and let* $\operatorname{span} V_{\vec{L}}[t] := \max V_{\vec{L}}[t] - \min V_{\vec{L}}[t]$. *Then* $\operatorname{span} V_{\vec{L}}[t] \leq n$*, with equality if $\vec{L}_\Pi$ is alternating and reduced.*

Proof. Let $\operatorname{span}\langle L_\Pi\rangle[x] := \max\langle L_\Pi\rangle[x] - \min\langle L_\Pi\rangle[x]$. Using Lemma 2.12, we have that $\operatorname{span}\langle L_\Pi\rangle[x] \leq 2n + 2|s_A| + 2|s_B| - 4$, where, as usual, $s_A \in \mathcal{S}_{L_\Pi}$ denotes the state of $L_\Pi$ in which all crossings are $A$ split, and $s_B \in \mathcal{S}_{L_\Pi}$ denotes the state of $L_\Pi$ in which all crossings are $B$ split. Note that when we multiply $\langle L_\Pi\rangle[x]$ by $(-x^3)^{-w(\vec{L}_\Pi)}$ to get the Kauffman polynomial, we increase both $\max\langle L_\Pi\rangle[x]$ and $\min\langle L_\Pi\rangle[x]$ by $-3w(\vec{L}_\Pi)$, and so their difference does not change. Thus, if we let $\operatorname{span} X_{\vec{L}}[x] := \max X_{\vec{L}}[x] - \min X_{\vec{L}}[x]$, then $\operatorname{span} X_{\vec{L}}[x] = \operatorname{span}\langle L_\Pi\rangle[x]$. Hence, $\operatorname{span} X_{\vec{L}}[x] \leq 2n + 2|s_A| + 2|s_B| - 4$. Since the substitution $x \mapsto t^{-\frac{1}{4}}$ scales all the exponents in the bracket polynomial by a factor of 4, we find that $\operatorname{span} V_{\vec{L}}[t] = \operatorname{span} X_{\vec{L}}[x] \div 4$, or, equivalently, $4 \cdot \operatorname{span} V_{\vec{L}}[t] = \operatorname{span} X_{\vec{L}}[x]$. Thus, $4 \cdot \operatorname{span} V_{\vec{L}}[t] \leq 2n + 2|s_A| + 2|s_B| - 4$. By Lemma 2.13, however, $2|s_A| + 2|s_B| \leq 2n + 4$, whence $4 \cdot \operatorname{span} V_{\vec{L}}[t] \leq 4n$, and so $\operatorname{span} V_{\vec{L}}[t] \leq n$.

If $\vec{L}_\Pi$ is alternating, connected, and reduced, then it is adequate (Lemma 2.11). Therefore, by Lemma 2.12, $\operatorname{span}\langle L_\Pi\rangle[x] = 2n + 2|s_A| + 2|s_B| - 4$. Moreover, because $\vec{L}_\Pi$ is alternating and connected, Lemma 2.13 tells us that $2|s_A| + 2|s_B| = 2n + 4$. As before, $4 \cdot \operatorname{span} V_{\vec{L}}[t] = \operatorname{span} X_{\vec{L}}[x] = \operatorname{span}\langle L_\Pi\rangle[x]$, and thus $\operatorname{span} V_{\vec{L}}[t] = n$. ■

**5.5. Step 5: Endgame.**

Proof of Theorem 2.5. Begin by giving an orientation to each of the components of $L_{\Pi_1}$ and $L_{\Pi_2}$. Then we can use $\vec{L}_{\Pi_1}$ to compute the Jones polynomial of $\vec{L}$. Indeed, because this projection is alternating, connected, and reduced, $\operatorname{span} V_{\vec{L}}[t] = n_1$, by Lemma 2.14. However, because $\vec{L}_{\Pi_2}$ has $n_2$ crossings (and is connected), Lemma 2.14 also says that $\operatorname{span} V_{\vec{L}}[t] \leq n_2$, and so $n_1 \leq n_2$, as desired. ■

## 6. Proof of the Second Tait Conjecture

The reader may very well have forgotten the precise statement of the Second Tait Conjecture, and so we paraphrase it here for convenience: any two alternating, reduced, and connected projections of an oriented link will have the same writhe. In fact, the proof will draw on many results from the previous section, including Theorem 2.5! As before, we will break apart the proof into several small steps.[10] The first two steps serve to introduce two new concepts, *linking numbers* and r-*parallels*. We begin with the former.

[10]Again, we will follow the proofs in Lickorish's text [**?**]; his methods, however, were first developed by Stong [**?**].

**6.1. Step 1: The Linking Number.** Recall that Reidemeister I moves always alter the writhe of an oriented link projection by $\pm 1$. Notice, however, that these moves introduce or remove crossings that involve *only one* of the link's components. Thus, if we *only* consider the crossings at which two *distinct* components of the link meet, we might stand a better chance at coming up with an invariant. These observations motivate the following definition.

DEFINITION 2.15. *Suppose that $\vec{L}_\Pi$ is a projection of an oriented link $\vec{L}$ with more than one component. Select any two components of $\vec{L}$; denote them as $\vec{L}^i$ and $\vec{L}^j$, and let $\vec{L}^i_\Pi$ and $\vec{L}^j_\Pi$ denote the pieces of $\vec{L}_\Pi$ to which they correspond. The* linking number *of $\vec{L}^i_\Pi$ and $\vec{L}^j_\Pi$, denoted $lk(\vec{L}^i_\Pi, \vec{L}^j_\Pi)$, is the sum, taken over crossings involving only strands from both $\vec{L}^i_\Pi$ and $\vec{L}^j_\Pi$, where each term is either* $1$ *or* $-1$, *depending on whether the crossing is of* $+1$ *type or* $-1$ *type.*[11]

EXAMPLE 2.16. For the oriented Hopf link $\vec{H}$, let $\vec{H}_\Pi$ denote the projection of $\vec{H}$ that is depicted on the right of Figure **??**. Each of the two crossings in $\vec{H}_\Pi$ involve distinct components, and so $lk(\vec{H}^1_\Pi, \vec{H}^2_\Pi) = 2$.

It is important to note that the linking number is defined for *pairs* of components in a link, and so any projection of a link with $n$ components has associated with it $\binom{n}{2}$ individual linking numbers.

We will now formally prove the invariance of a linking number.

PROPOSITION 2.17. *Suppose that $\vec{L}_{\Pi_1}$ and $\vec{L}_{\Pi_2}$ are two projections of an oriented link $\vec{L}$. Select any two components of $\vec{L}$; denote them as $\vec{L}^i$ and $\vec{L}^j$. Let $\vec{L}^i_{\Pi_1}$ and $\vec{L}^i_{\Pi_2}$ denote the pieces of $\vec{L}_{\Pi_1}$ and $\vec{L}_{\Pi_2}$, respectively, to which $\vec{L}^i$ corresponds. Let $\vec{L}^j_{\Pi_1}$ and $\vec{L}^j_{\Pi_2}$ denote the pieces of $\vec{L}_{\Pi_1}$ and $\vec{L}_{\Pi_2}$, respectively, to which $\vec{L}^j$ corresponds. Then $lk(\vec{L}^i_{\Pi_1}, \vec{L}^j_{\Pi_1}) = lk(\vec{L}^i_{\Pi_2}, \vec{L}^j_{\Pi_2})$.*

PROOF. Because $\vec{L}_{\Pi_1}$ and $\vec{L}_{\Pi_2}$ are two projections of the same oriented link, there is a finite sequence of Reidemeister moves and planar isotopies that transforms $\vec{L}_{\Pi_1}$ into $\vec{L}_{\Pi_2}$. Clearly, planar isotopies will not effect the linking number. Moreover, the proof of Proposition 1.17 can apply equally well in this situation (assuming, of course, that the strands involved belong to different components) to show that $lk(\vec{L}^i_{\Pi_1}, \vec{L}^j_{\Pi_1}) = lk(\vec{L}^i_{\Pi_2}, \vec{L}^j_{\Pi_2})$ when the two projections differ by a Reidemeister II or III move. Hence, in each of the possible cases, $lk(\vec{L}^i_{\Pi_1}, \vec{L}^j_{\Pi_1}) = lk(\vec{L}^i_{\Pi_2}, \vec{L}^j_{\Pi_2})$, as desired. ■

Proposition 2.17, in turn, motivates the following definition.

DEFINITION 2.18. *Suppose that $\vec{L}$ is a link with more than one component. Select any two components of $\vec{L}$; denote them by $\vec{L}^i$ and $\vec{L}^j$. Let $\vec{L}_\Pi$ be any projection of $\vec{L}$, and let $\vec{L}^i_\Pi$ and $\vec{L}^j_\Pi$ denote the pieces of $\vec{L}_\Pi$ to which $\vec{L}^i$ and $\vec{L}^j$ correspond. Then the* linking number *of $\vec{L}^i$ and $\vec{L}^j$, denoted by $lk(\vec{L}^i, \vec{L}^j)$, is given by $lk(\vec{L}^i, \vec{L}^j) := lk(\vec{L}^i_\Pi, \vec{L}^j_\Pi)$.*

Again, we reiterate that a link with $n$ components has $\binom{n}{2}$ linking numbers naturally associated with it.

[11] This formulation is slightly nonstandard; usually the sum is divided by $\frac{1}{2}$.

Linking numbers will resurface again in §2.6.3. For the time being, however, we put them aside, and turn to the concept of $r$-parallels.

**6.2. Step 2: *R*-Parallels.** Much attention has been given in this thesis to the notion of link projections being as simple as possible. Therefore, it may surprise the reader that our proof of Theorem 2.6 will involve an intermediate passage to an objectively *more* complicated class of link projections. Nonetheless, the approach, which uses r-*parallels*, is quite clever. The relevant definition is as follows.

DEFINITION 2.19. *Suppose that $L_\Pi$ is a projection of a link $L$. For a positive integer $r$, the* r-parallel *of $L_\Pi$, denoted $(L_\Pi)^r$, is a projection (of a new link) in which each strand of $L_\Pi$ is joined by $r-1$ parallel copies of that strand; as a collection, the $r$ strands follow the crossing patterns of $L_\Pi$.*

EXAMPLE 2.20. The picture on the left of Figure **??** depicts the 2-parallel of the trefoil knot projection, and the picture on the right depicts the 2-parallel of the Hopf link projection.

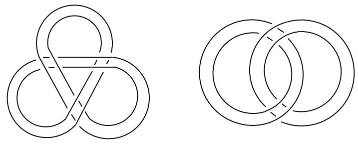

FIGURE 29. The 2-parallels of the trefoil knot projection (left) and the Hopf link projection (right).

Several preliminary remarks regarding $r$-parallels are in order. First, if a projection is oriented, then all of its parallels are (by convention) given the same orientation. Second, note from Figure **??** that $r$-parallels are not alternating! Still, it makes sense to talk about the plus- or minus-adequacy, and luckily, taking $r$-parallels does not destroy that property; we formalize this with a lemma.

LEMMA 2.21. *Suppose that $L_\Pi$ is a projection of a link $L$. Let $(L_\Pi)^r$ denote the $r$-parallel of $L_\Pi$. If $L_\Pi$ is plus-adequate, then $L_\Pi^r$ is plus-adequate. Moreover, if $L_\Pi$ is minus-adequate, then $(L_\Pi)^r$ is minus-adequate.*

PROOF. We treat the case when $L_\Pi$ is plus-adequate; the case when $L_\Pi$ is minus-adequate is identical. Let $s_A \in \mathcal{S}_{(L_\Pi)^r}$ denote the state of $L_\Pi^r$ in which all crossings are $A$ split. Let $s'_A \in \mathcal{S}_{L_\Pi}$ denote the state of $L_\Pi$ in which all crossings are $A$ split. Referring to Figure **??**, we note that, after performing some planar isotopies to smooth out the wrinkles, $s'_A = (s_A)^r$ (i.e., to each component of the unlink in $s_A$, we add $r-1$ parallel copies). Recalling the test for plus-adequacy, we must check that none of the components of $s'_A$ meets itself at the site of a former crossing; of course, there are now many more crossings to consider. However, the geometry of the original link projection comes to our aid. Indeed, because no component of $s_A$ meets itself, and because the components of $s'_A$ can be thought of as running parallel to the original components of $s_A$, it becomes geometrically clear that no component of $s'_A$ can meet itself. Hence, $(L_\Pi)^r$ is plus-adequate. ∎

The reason for introducing $r$-parallels into our toolkit will become apparent momentarily.

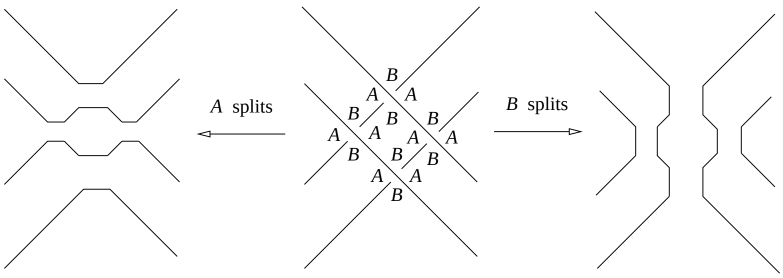


FIGURE 30. Performing all $A$ splits (left) and all $B$ splits (right) on a generic crossing of a 2-parallel.

**6.3. Step 3: The Main Lemma.** The following important lemma relates the number of crossings in plus-adequate diagrams to their writhes, and ties together the concepts of linking numbers and $r$-parallels; moreover, it incorporates one of our link polynomials. Theorem 2.6 will follow directly from this lemma.

LEMMA 2.22. *Suppose that $\vec{L}_{\Pi_1}$ and $\vec{L}_{\Pi_2}$ are two projections of an oriented link $\vec{L}$. Suppose that $\vec{L}_{\Pi_1}$ is plus-adequate. Let $n_1$ and $n_2$ denote, respectively, the number of crossings in $\vec{L}_{\Pi_1}$ and $\vec{L}_{\Pi_2}$. Let $w(\vec{L}_{\Pi_1})$ and $w(\vec{L}_{\Pi_2})$ denote, respectively, the writhes of $\vec{L}_{\Pi_1}$ and $\vec{L}_{\Pi_2}$. Then $n_1 - w(\vec{L}_{\Pi_1}) \leq n_2 - w(\vec{L}_{\Pi_2})$.*

PROOF. Index each of the components of $\vec{L}$ as $\vec{L}^1, \vec{L}^2, \dots \vec{L}^m$. Then, each $\vec{L}_i$ corresponds to some piece of $\vec{L}_{\Pi_1}$, and some piece of $\vec{L}_{\Pi_2}$. Let the piece of $\vec{L}_{\Pi_1}$ and $\vec{L}_{\Pi_2}$ that correspond to $\vec{L}^i$ be denoted as $\vec{L}^i_{\Pi_1}$ and $\vec{L}^i_{\Pi_2}$.

For all $i$, $\vec{L}^i_{\Pi_1}$ and $\vec{L}^i_{\Pi_2}$ have well-defined writhes. To compute $w(\vec{L}^i_{\Pi_k})$ for a given $i$, however, we will (for the time being) *disregard* the crossings in $\vec{L}^i_{\Pi_k}$ that involve other components; in so doing, we are treating each $\vec{L}^i_{\Pi_k}$ as a link projection in its own right by imagining that the other components simply are not there. With that qualification understood, we can choose, for each $i$, nonnegative integers $d_i$ and $e_i$ such that $w(\vec{L}^i_{\Pi_1}) + d_i = w(\vec{L}^i_{\Pi_2}) + e_i$. In order to realize this equality geometrically (i.e., in order to make the writhes of $\vec{L}^i_{\Pi_1}$ and $\vec{L}^i_{\Pi_2}$ equal), we can add very tiny positive twists (i.e., twists which introduce crossings whose type, in the sense of §1.3, is $+1$) by performing appropriate Reidemeister I moves (see the diagram on the right-hand side of Figure **??**). The tininess ensures that the new twists do not interact with the other strands. By performing this process to each $\vec{L}^i_{\Pi_k}$, we have added $\sum_i d_i$ and $\sum_i e_i$ positive twists to $\vec{L}_{\Pi_1}$ and $\vec{L}_{\Pi_2}$, respectively. Call these new link diagrams $\vec{L}^{\times}_{\Pi_1}$ and $\vec{L}^{\times}_{\Pi_2}$.

Adding the positive twists clearly does not change the plus-adequacy of $\vec{L}_{\Pi_1}$. Thus, $\vec{L}^{\times}_{\Pi_1}$ is also plus-adequate. Now we compare $w(\vec{L}^{\times}_{\Pi_1})$ with $w(\vec{L}^{\times}_{\Pi_2})$. The effect of adding the positive twists was to make $w(\vec{L}^i_{\Pi_1}) = w(\vec{L}^i_{\Pi_2})$ for all $i$. However, recall that, in computing $w(\vec{L}^i_{\Pi_k})$, we *ignored* those crossings which involved distinct components. But such crossings are precisely the sites used in the computation of the various linking numbers. Indeed, we can think of the writhe of an oriented link as being computed using two sets of crossings: those where components cross themselves, and those where distinct components cross each other. Symbolically, then, we have:

$$w(\vec{L}^{\times}_{\Pi_k}) = \sum_i w(\vec{L}^{i}_{\Pi_k}) + \sum_i \{\begin{smallmatrix} d_i \text{ if } k=1 \\ e_i \text{ if } k=2 \end{smallmatrix}\} + \sum_{i<j} lk(\vec{L}^i, \vec{L}^j)$$

Indeed, since linking numbers are invariant, it follows that $w(\vec{L}^{\times}_{\Pi_1}) = w(\vec{L}^{\times}_{\Pi_2})$.

Consider now the $r$-parallel $(\vec{L}^{\times}_{\Pi_1})^r$ and $(\vec{L}^{\times}_{\Pi_2})^r$. Clearly, they are both projections of the same link ($\vec{L}$, with $r-1$ parallel components added) and hence are equivalent. Therefore, they have the same Kauffman polynomial. Moreover, since every crossing of $\vec{L}^{\times}_{\Pi_1}$ and $\vec{L}^{\times}_{\Pi_2}$ corresponds to $r^2$ crossings of $(\vec{L}^{\times}_{\Pi_1})^r$ and $(\vec{L}^{\times}_{\Pi_2})^r$, we see that $w((\vec{L}^{\times}_{\Pi_1})^r) = r^2 w(\vec{L}^{\times}_{\Pi_1}) = r^2 w(\vec{L}^{\times}_{\Pi_2}) = w((\vec{L}^{\times}_{\Pi_2})^r)$. By the definition of the Kauffman polynomial, it follows immediately that $\langle(\vec{L}^{\times}_{\Pi_1})^r\rangle = \langle(\vec{L}^{\times}_{\Pi_2})^r\rangle$.

Let $|s_{1,A}|$ denote the number of connected components in an all-$A$ splitting of $\vec{L}_{\Pi_1}$, and let $|s_{2,A}|$ denote the number of connected components in all-$A$ splitting of $\vec{L}_{\Pi_2}$. Adding the positive twists to $\vec{L}_{\Pi_1}$ and $\vec{L}_{\Pi_2}$ means that the number of connected components in the all-$A$ splitting of $\vec{L}^{\times}_{\Pi_1}$ and $\vec{L}^{\times}_{\Pi_2}$ becomes $|s_{1,A}| + \sum_i d_i$ and $|s_{2,A}| + \sum_i e_i$. Moreover, when we pass to the $r$-parallels, we find that the number of connected components in the all-$A$ splitting of $(\vec{L}^{\times}_{\Pi_1})^r$ and $(\vec{L}^{\times}_{\Pi_2})^r$ becomes $r(|s_{1,A}| + \sum_i d_i)$ and $r(|s_{2,A}| + \sum_i e_i)$.

Likewise, adding the positive twists to $\vec{L}_{\Pi_1}$ and $\vec{L}_{\Pi_2}$ means that the number of crossings in $\vec{L}^{\times}_{\Pi_1}$ and $\vec{L}^{\times}_{\Pi_2}$ becomes $n_1 + \sum_i d_i$ and $n_2 + \sum_i e_i$. Furthermore, making $r$-parallels means that the number of crossings in $(\vec{L}^{\times}_{\Pi_1})^r$ and $(\vec{L}^{\times}_{\Pi_2})^r$ becomes $(n_1 + \sum_i d_i)r^2$ and $(n_2 + \sum_i e_i)r^2$.

Since $\vec{L}^{\times}_{\Pi_1}$ is plus-adequate (discussed above), we have, by Lemma 2.21, that $(\vec{L}^{\times}_{\Pi_1})^r$ is also plus-adequate. By Lemma 2.12:

$$\begin{aligned} \max\langle(\vec{L}^{\times}_{\Pi_1})^r\rangle &= 2(r(|s_{1,A}| + \sum_i d_i) - 1) + (n_1 + \sum_i d_i)r^2 \\ &= (n_1 + \sum_i d_i)r^2 + 2(|s_{1,A}| + \sum_i d_i)r - 2 \end{aligned}$$

Also by Lemma 2.12:

$$\begin{aligned} \max\langle(\vec{L}^{\times}_{\Pi_2})^r\rangle &\leq 2(r(|s_{2,A}| + \sum_i e_i) - 1) + (n_2 + \sum_i e_i)r^2 \\ &= (n_2 + \sum_i e_i)r^2 + 2(|s_{2,A}| + \sum_i e_i)r - 2 \end{aligned}$$

Since $\langle(\vec{L}^{\times}_{\Pi_1})^r\rangle = \langle(\vec{L}^{\times}_{\Pi_2})^r\rangle$, $\max\langle(\vec{L}^{\times}_{\Pi_1})^r\rangle = \max\langle(\vec{L}^{\times}_{\Pi_2})^r\rangle$, and thus:

$$\begin{aligned} (n_1 + \sum_i d_i)r^2 + 2(|s_{1,A}| + \sum_i d_i)r - 2 &\leq (n_2 + \sum_i e_i)r^2 + 2(|s_{2,A}| + \sum_i e_i)r - 2 \\ (n_1 + \sum_i d_i)r^2 + 2(|s_{1,A}| + \sum_i d_i)r &\leq (n_2 + \sum_i e_i)r^2 + 2(|s_{2,A}| + \sum_i e_i)r \end{aligned}$$

an inequality which must be true for *all* positive integers $r$. Taking $r$ to be very large, we find, upon comparing coefficients of $r^2$, that $n_1 + \sum_i d_i \leq n_2 + \sum_i e_i$. Recalling that $w(\vec{L}^i_{\Pi_1}) + d_i = w(\vec{L}^i_{\Pi_2}) + e_i$, we have:

$$-\sum_i w(\vec{L}^i_{\Pi_1}) - \sum_i d_i = -\sum_i w(\vec{L}^i_{\Pi_2}) - \sum_i e_i$$

and adding this to $n_1 + \sum_i d_i \leq n_2 + \sum_i e_i$ gives that $n_1 - \sum_i w(\vec{L}^i_{\Pi_1}) \leq n_2 - \sum_i w(\vec{L}^i_{\Pi_2})$. Again, using the fact that the linking number is an invariant, we can subtract $lk(\vec{L}^i_{\Pi_1}, \vec{L}^j_{\Pi_1}) = lk(\vec{L}^i_{\Pi_2}, \vec{L}^j_{\Pi_2})$ from both sides of this equation, one for each pair of distinct components of $\vec{L}$, thereby transforming $\sum_i w(\vec{L}^i_{\Pi_1})$ and $\sum_i w(\vec{L}^i_{\Pi_2})$ into $w(\vec{L}_{\Pi_1})$ and $w(\vec{L}_{\Pi_2})$. Thus, $n_1 - w(\vec{L}_{\Pi_1}) \leq n_2 - w(\vec{L}_{\Pi_2})$, as desired. ■

### 6.4. Step 4: Endgame.

Proof of Theorem 2.6. Let $n_1$ and $n_2$ denote, respectively, the number of crossings in $\vec{L}_{\Pi_1}$ and $\vec{L}_{\Pi_2}$. Because $\vec{L}_{\Pi_1}$ and $\vec{L}_{\Pi_2}$ are alternating, reduced, and connected projections, they are adequate (and hence plus-adequate). Thus, by Lemma 2.22, $n_1 - w(\vec{L}_{\Pi_1}) \leq n_2 - w(\vec{L}_{\Pi_2})$ and $n_2 - w(\vec{L}_{\Pi_2}) \leq n_1 - w(\vec{L}_{\Pi_1})$, and so $n_1 - w(\vec{L}_{\Pi_1}) = n_2 - w(\vec{L}_{\Pi_2})$. Moreover, since alternating, connected, and reduced projections of links have minimal crossing number (Theorem 2.5), $n_1 = n_2$, and thus $w(\vec{L}_{\Pi_1}) = w(\vec{L}_{\Pi_2})$ ■

Corollary 2.23. *Suppose that $L_{\Pi_1}$ and $L_{\Pi_2}$ are both alternating, connected, and reduced projections of an oriented link $\vec{L}$. Then the number of $+1$ crossings in each of $L_{\Pi_1}$ and $L_{\Pi_2}$ is the same, and similarly for the number of $-1$ crossings.*

Proof. This follows at once from Theorem 2.5 and Theorem 2.6. ■

Not only do the proofs of Theorems 2.5 and 2.6 demonstrate the power of polynomial invariants, but they also underscore the fruitfulness of using states to analyze links.

## 7. Applications of the Tait Conjectures

We will now describe some straightforward (and pretty!) applications of the Tait Conjectures.

**7.1. Amphicheirality.** Imagine that we take our standard projection of the trefoil knot and *reverse* each of its crossings (see Figure **??**). This is known as taking the *mirror image* of a link projection.

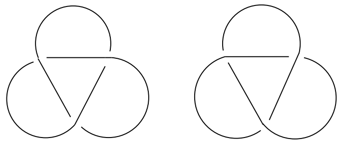

FIGURE 31. Our standard projection of the trefoil knot (left) and its mirror image (right).

A natural question to ask, then, is whether or not we can transform the trefoil knot projection into its mirror image using Reidemeister moves and/or planar isotopies. In general, if a given link projection can be deformed into its mirror image, we say that the underlying link is *amphicheiral.* In fact, if the link projection is alternating, connected, and reduced, we can use Theorem 2.6 to deduce some necessary conditions for amphicheirality. The theorem is as follows.

THEOREM 2.24. *Suppose that $L_\Pi$ is an alternating, connected, and reduced projection of a link $L$. Let $\overline{L_\Pi}$ denote the mirror image of $L_\Pi$ If the number of crossings in $L_\Pi$ is odd, then $L$ can never be amphicheiral.*

PROOF. When we reverse each crossing, we change its type; hence, $w(\overline{L_\Pi}) = -w(L_\Pi)$. Suppose now that $L_\Pi$ is equivalent to $\overline{L_\Pi}$. By assumption, $\overline{L_\Pi}$ will *also* be alternating and reduced. Therefore, because $L_\Pi$ and $\overline{L_\Pi}$ are both alternating and reduced projections of the same link, they must, by Theorem 2.6, have the same writhe; thus, $w(L_\Pi) = w(\overline{L_\Pi}) = -w(L_\Pi)$, and so it follows that $w(L_\Pi) = 0$. However, the writhe of a link can *only* equal zero if it has an even number of crossings, and that gives the theorem. ■

Hence, because our alternating, reduced projection of the trefoil knot has an odd number of crossings, it *cannot* be transformed into its mirror image through any conceivable combination of Reidemeister moves and planar isotopies! As it turns out, the knot depicted on the left in Figure **??** — known as the *figure-eight knot* — is amphicheiral; this can be verified directly without too much difficulty.

**7.2. The Minimal Crossing Number of Composite Knots.** Given two knots $K_1$ and $K_2$, their *composite* — also sometimes referred to as their *connected sum* — is a new knot, denoted $K_1\#K_2$, which is formed by cutting a very tiny section of string from each of $K_1$ and $K_2$, and then splicing the two knots together; the knots $K_1$ and $K_2$ are called the *factor knots* of $K_1\#K_2$. For example, Figure **??** depicts the construction of the composite of the figure-eight knot and the trefoil knot.[12]

[12]There are some qualifications that need to be made about where on the projection we are allowed to perform this surgery; technically, we can only splice knots on their outer-most strands.

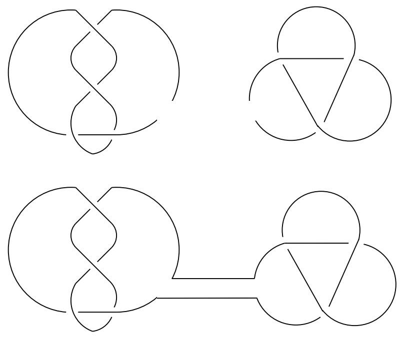

FIGURE 32. Forming the composite of the figure-eight knot and the trefoil knot.

One might wonder whether or not it is possible to redraw $K_1\#K_2$ so that it has *fewer* than 7 crossings. It turns out that the answer is no, and the reason has everything to do with the fact that $K_1\#K_2$ is *also* an alternating knot.

Indeed, William Menasco proved a theorem which says that if $(K_1\#K_2)_\Pi$ is an alternating projection of a knot $K_1\#K_2$, then there is a topological loop in the projection plane that intersects $(K_1\#K_2)_\Pi$ exactly twice, so that the factor knots that appear on either side of the loop are alternating (see, for example, Figure **??**).[13]

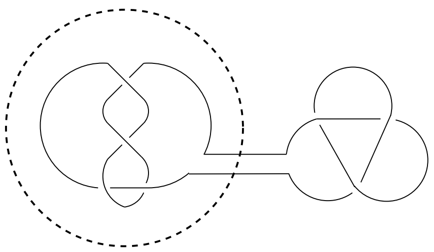

FIGURE 33. A special case of Menasco's theorem.

Combining Menasco's Theorem and Theorem 2.5, we obtain the following result.

THEOREM 2.25. *Suppose that $K_1\#K_2$ is an alternating knot. Then the minimal number of crossings that can be realized in a projection of $K_1\#K_2$ equals the sum of the minimal number of crossings that can be realized in projections of $K_1$ and $K_2$.*

PROOF. Choose a reduced, alternating projection $(K_1\#K_2)_\Pi$ of $K_1\#K_2$. By Menasco's theorem, the factor knots $K_1$ and $K_2$ will appear alternating in $(K_1\#K_2)_\Pi$. Since $(K_1\#K_2)_\Pi$ is reduced, so are the projections of the factor knots in $(K_1\#K_2)_\Pi$. Therefore, we can apply Theorem 2.5 to conclude that the factor knots, being alternating and reduced (and obviously connected), are each drawn with the minimum possible number of crossings. Moreover, because $(K_1\#K_2)_\Pi$ is alternating and reduced (and obviously connected), it too is drawn with the minimum possible number of crossings. But the number of crossings that appears in $(K_1\#K_2)_\Pi$ is clearly equal to the number of crossings that appears in each of the factor knots, which gives the theorem. ■

[13] We are stating the theorem as it is found on page 162 of Adams' book [**?**].

Theorem 2.25 tells us, then, that the composite knot shown in Figure **??** cannot be drawn with less than 7 crossings. It is still an open question whether or not the minimal number of crossings that can be realized in a projection of $K_1 \# K_2$ for *non*-alternating, reduced knots $K_1$ and $K_2$ equals the sum of the minimal number of crossings that can be realized in projections of $K_1$ and $K_2$.

## 8. The Third Tait Conjecture

We have, at this point, said all we wish to say about states, link polynomials, and the First and Second Tait Conjectures. To close, we will briefly describe the *Third* Tait Conjecture, whose existence we alluded to in §2.4. Although its stock-in-trade is the same — alternating, connected, and reduced link projections — its conclusion is slightly less intuitive.

Given any link, we can easily draw a circle in the projection plane that encloses some region of the link. If, however, the circle intersects the link *exactly* four times, we call the enclosed region a *tangle*. Examples of tangles are given below in Figure **??**.

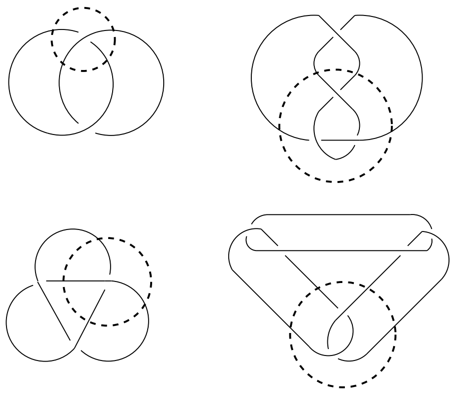

FIGURE 34. Examples of tangles.

There is a special transformation, involving tangles, that can be performed on links: we simply choose any tangle in the link, fix the four points of the link that lie on the circle, and then rotate the tangle by 180°. This operation is called a *flype*,[14] and it is illustrated in Figure **??**. Notice how flypes will take a crossing one side of a tangle and move it to the other side.

[14]According to `www.mathworld.com`, "flype" derives from the Scottish verb meaning "to turn or fold back."

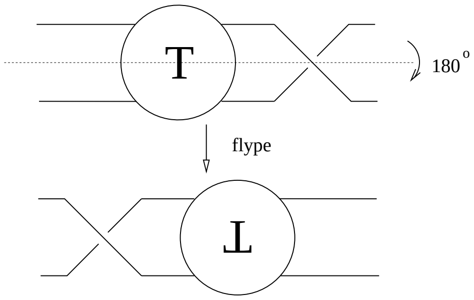


FIGURE 35. Performing a flype.

Tait conjectured that any two reduced, alternating projections of the same knot can be transformed into each other through a finite sequence of flypes. (In fact, the conjecture is true more generally for alternating, connected, and reduced projections of the same link.) Like its sisters, the Third Tait Conjecture remained an open problem until the discovery of the Jones polynomial, which is used in its proof. However, the techniques for proving it are *much* different than those employed in this thesis; indeed, they merit a thesis all to themselves.

* * *

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