FANO VARIETIES OF LOW-DEGREE SMOOTH HYPERSURFACES AND UNIRATIONALITY

AUTHOR: ALEX WALDRON

ADVISOR: JOE HARRIS

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Harvard University
Cambridge, Massachusetts
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[1] *Email:*
waldron@fas.harvard.edu
harris@math.harvard.edu

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Introduction.

Given a projective variety $X \subset \mathbb{P}^n$ and an integer $k > 0$, we ask a classical question: does X contain a projective linear space of dimension k?

As it stands, this question invites the following explicit line of attack. Let $\{F_\alpha\}$ be a set of homogeneous polynomials defining X. Parametrize a k-plane Λ via a linear map $\mathbb{P}^k \to \Lambda \subset \mathbb{P}^n$, and consider the expressions $F_\alpha(\Lambda)$—these are homogeneous polynomials in the parameters of Λ. Then $\Lambda \subset X$ if and only if each $F_\alpha(\Lambda)$ vanishes identically with respect to the parameters. But the coefficients of the polynomials $F_\alpha(\Lambda)$ are themselves polynomials $C_{\alpha\beta}$ on the space of linear maps $\mathbb{P}^k \to \mathbb{P}^n$. Taking slightly greater care with these parametrizations (Remark 1.2 below), we can use elimination theory (see [18] Ch. 14, 15) to determine computationally whether or not the equations $C_{\alpha\beta} = 0$ have a simultaneous solution that corresponds to a k-plane $\Lambda \subset X$. Our initial question is thus answered for the given variety X, provided we can successfully perform these computations over our base field K.

Such computational methods are not, however, the subject of this paper. Rather, we hope to determine general circumstances under which we can expect to find k-planes on varieties. And the previous argument suggests the point of entry for the full techniques of algebraic geometry: the set of k-planes contained in a given X form a projective variety, embeddable in the Grassmannian $\mathbb{G}(k, n)$, which we shall call the *Fano Variety* $F_k(X) \subset \mathbb{G}(k, n)$. This observation allows us to rephrase the initial question quantitatively: what is the *dimension* of the variety of k-planes $F_k(X)$ lying on X? If we can establish that $\dim(F_k(X)) \geq 0$, then we can answer the original question to the affirmative.

In this paper, we will restrict our attention to the natural first case, that of hypersurfaces —this will allow us, crucially, to consider all hypersurfaces of degree d in $\mathbb{P}^n$ simultaneously via their parameter space $\mathbb{P}^N$. With this advantage, we will be able to give several answers that in fact depend on no more than this degree d as compared to the dimensions k and n, to which we will eventually add the requirement of smoothness.

0.1. **Dimension of Fano varieties.** Our first and most general claim concerning k-planes on hypersurfaces is as follows. Define

$$\phi(n, d, k) = (k + 1)(n - k) - \binom{k + d}{d}.$$

Let $X \subset \mathbb{P}^n$ be a hypersurface of degree $d \geq 3$. We claim that

$$\dim(F_k(X)) \geq \phi(n, d, k) \text{ if } \phi \geq 0. \tag{1}$$

For a general X, we claim that (1) is an equality if $\phi \geq 0$, and that

$$F_k(X) = \emptyset \text{ if } \phi < 0.$$

If this holds for a particular X, we will say that $F_k(X)$ has "the expected dimension."

Restated, our claim is that for a general hypersurface X,

$$\operatorname{codim}(F_k(X) \subset \mathbb{G}(k, n)) = \binom{k + d}{d} = \#\{\text{degree } d \text{ monomials on } \mathbb{P}^k\}.$$

In the crude argument given above, this is the number of coefficients "C_β" of "$F(\Lambda)$," the restriction to Λ of a defining polynomial F of the hypersurface X. So, this would appear to be the number of conditions cutting out $F_k(X)$. However $F_k(X)$ is a subvariety of the Grassmannian $\mathbb{G}(k,n)$, and therefore such an argument fails to show that $F_k(X)$ is in fact non-empty if $\phi \geq 0$. Indeed, the requirement $d \geq 3$ is essential, as shown in Remark 1.8. And, although the argument will be entirely classical—a dimension count via incidence correspondences (two, in this case)—the claim was established only through the result of Hochster and Laksov [1] in 1987.

Interestingly, in the case $\phi(n,d,k) \geq 0$, the proof that $F_k(X)$ is non-empty *for all* $X \subset \mathbb{P}^n$ of a given degree d will depend on proving the *existence* of a hypersurface X_0 such that $F_k(X_0)$ has dimension *exactly* $\phi(n,d,k)$ (see Remark 1.9). We will give several such examples, thereby proving particular cases of the claim. For arbitrary n,d,k, we will prove that a general surface has Fano variety of dimension ϕ; even while we are left with no way to exhibit such a hypersurface, nor with means to check that a particular hypersurface is "general" in our sense. This is the power of using "incidence correspondences" (section 1.2.1) in conjunction with the Theorem on Fiber Dimension (Proposition 1.3), in contrast to the naive computational argument suggested at the outset.

0.2. **The low-degree limit.** This last result establishes that *all* hypersurfaces have "enough" k-planes corresponding to their degree and dimension, and that a general hypersurface has the expected-dimensional family of k-planes. But the next question remains inscrutable: which hypersurfaces have "too many" k-planes, by which we mean a Fano variety of dimension greater than ϕ? We propose two criteria for remedying this situation, i. e. specifying which hypersurfaces are "general" in the previous sense. The first, smoothness, is natural especially when working over a field K of characteristic zero (as we will in Ch. 2 and 3). However, it is not sufficient. In the case $k=1$ of lines, the canonical example of a smooth hypersurface fails: if $\mathrm{char}(K)=0$, the Fano variety of the Fermat hypersurface of degree $d=n+1$ in $\mathbb{P}^n$ has dimension $n-3$ (see [5]), greater than the estimated dimension $\phi(n,d,1)=2n-3-d=n-4$.

Second, we work in the limit of low degree compared to dimension. This is a realm of broad interest pertaining to several different fields: see for example Kollar's article [12]. By itself, however, low degree is clearly insufficient for $F_k(X)$ to have the expected dimension, as seen by considering any reducible hypersurface.

Working in the low-degree limit is a common approach for "specializing" a known fact about general hypersurfaces to smooth ones.[2] In the case of lines, $k=1$, the well-known Debarre-De Jong conjecture [3] asserts that if $d \leq n$ and X is smooth, then $\dim(F_k(X)) = \phi(n,d,1) = 2n-3-d$, i. e. X does not have "too many" lines. By the example just given, the bound $d \leq n$ is sharp, if it holds. This conjecture is actively pursued (see [3], [4]): the result has been established for $d \leq 6$ [5], and for several months in 2007 a proof of the general case was thought to have been found. There has also been progress in the area of rational curves lying on hypersurfaces of low degree: Starr and Harris [10] show that for $d < (n+1)/2$, a general degree d hypersurface contains the expected-dimensional variety of rational curves of each degree.

Our assertion is that for a fixed k, *if* $X \subset \mathbb{P}^n$ *is smooth of degree* $d \ll n$, *then* X *does not have too many* k*-planes.* This is the result of Harris, Mazur and Pandharipande [2] in 1998, in which these same Fano varieties are also shown to be irreducible. The proof

[2]The term "specialization" can refer to a much more involved set of techniques having this function.

will be by induction, using a more delicate incidence correspondence and the technique of residual intersections.

0.3. **Application to unirationality in low degree.** Finally, we will turn to a question not obviously related to the previous two, that of unirationality. A variety X is said to be *unirational* if there exists a dominant rational map $\mathbb{P}^N \to X$ for some N—or, equivalently, if the function field $K(X)$ is embeddable in a purely transcendental extension of the base field K. This property is a weakening of the important classical idea of rationality; but the latter notion has turned out to behave quite irrationally, even if we take $\mathrm{char}(K) = 0$. All irreducible quadrics are rational ([13] 7.14). Elliptic curves are not, but smooth cubic hypersurfaces in $\mathbb{P}^3$ are *always* rational. There are examples of smooth, cubic, rational projective hypersurfaces in all even dimensions, and there are known to exist the same in odd dimensions; but the general behavior in degree 3 is not known. Moreover, there are no smooth projective hypersurfaces of any degree $d \geq 4$ that are known to be rational, nor has the possibility been ruled out (for much of this, see [13]).

It was also long unknown whether unirationality is really a weaker notion than that of rationality for $\mathrm{char}(K) = 0$: a curve is unirational if and only if rational, as is a surface, as was shown by Castelnuovo and Enriques. (Whereas if $\mathrm{char}(K) = p$, the Zariski surfaces are unirational but not rational [8].) In 1972, however, Clemens and Griffiths showed that most cubic threefolds in $\mathbb{P}^4$ are not rational—we will prove in Proposition 2.3 that smooth ones are unirational, and thus that unirationality is a strictly weaker notion in characteristic zero (see [13] Ch. 7).

In contrast to the state of affairs concerning rationality, we will be able to show that *a smooth hypersurface in* $\mathbb{P}^n$ *of degree* $d \ll n$ *is unirational.* This theorem has its roots in the same assertion regarding a *general* hypersurface, dating back to Morin in 1940 [7]. In 1992, Paranjape and Srinivas [6] clarified Morin's proof and showed further that the general complete intersection of low multi-degree is unirational. Our result is again that of [2]: just as in the above result about Fano varieties, the unirationality result specializes to smooth hypersurfaces the earlier result for general ones. The construction of a rational "comb" of nested Grassmann bundles parametrizing the variety unirationally is similar to that of the earlier work, but in fact the crucial result above concerning Fano varieties allows the proof to hinge around smoothness.

0.4. **Further work.** We have yet to specify how we shall define "low degree." The following table shows us the meaning of $n \gg d$. (See sections 2.5 and 3.2.) These numbers serve as follows: let X be a smooth hypersurface of degree d in $\mathbb{P}^n$. If $n \geq N_0(d, k)$ then X does not have "too many" k-planes, and if $n \geq U(d)$ then X is unirational.

Judging by these high values,[3] the results of this paper (from [2]) might be deemed a bit quixotic. The bound $U(d)$ goes roughly as a $2d$-fold iterated exponential of d! Admittedly, the greatest care has not been taken to find the minimum such bounds attainable by our inductive methods. Still, these are the first and remain the best known bounds of their kind for arbitrary d and k. However, following Harris et. al. [2], Jason Starr [10] has recently reduced two very closely related bounds to a combinatorial expression in d and k.

[3]A typo in the original paper [2] was discovered subsequent to these computations—these values should actually be slightly higher.

TABLE 1. The meaning of $n \gg d$.

d	$N_0(d,k)$				$U(d)$
	$k=1$	$k=2$	$k=3$	$\cdots$	
2	4	6	9	$\cdots$	0
3	52	117	250		3
4	34276	366272	3391294		179124155
5	10^{17}	10^{21}	10^{25}	$\cdots$	10^{145}
6	10^{82}	10^{103}	10^{122}		10^{8790}
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$

Set $M(d,k) := \binom{k+d-1}{d-1} + k - 1$. (This number will reappear in Lemma 2.11, for the reason that if $n \geq M(d,k)$ then X is swept out by k-planes.) In [9], it is shown using much more sophisticated techniques that if $n \geq M(d,k)+1$ and X is smooth, then $F_k(X)$ has *at least a component* of the expected dimension. It is also shown there that if $n \geq M(d,k)$ then the general k-plane section of X is a general hypersurface in $\mathbb{P}^k$, whereas in [2] this same fact was proved only for $n > N_0(d-1,k)$. So [9] has managed to improve one of the main results of our source [2] to a practical level, leaving one optimistic about such prospects for the results which we demonstrate here.

Chen [11] has also shown as a corollary to [2] that the Fano variety is itself unirational in the low-degree limit.

0.5. **Incidence correspondences.** Perhaps the main idea of this paper is to take an "incidence correspondence" (see section 1.2.1) between two families in order to gain information about one by means of the other (often using the Theorem on Fiber Dimension (1.3)). An apology is even necessary for the repeated use of this idea in all three chapters—even the unirationally parametrizing variety will be built using the "relative Fano variety," which is locally just a form of the incidence correspondence "I" of Chapter 1!

Moreover, an idea similar to that of the first chapter is applied in the inductive step of our proof of unirationality: in Proposition 3.12 of Chapter 3, we will prove surjectivity by showing that the generic fiber of a map has appropriately low dimension, based on the fact from Ch. 2 that smooth hypersurfaces of low degree do not contain "too many" k-planes. (Although in Ch. 1 this argument is made for the projection onto the opposite factor.) The ubiquity of this construction becomes impressive.

1. Planes on Hypersurfaces in General.

We show that a general hypersurface X of degree $d \geq 3$ has Fano variety of dimension $\dim(F_k(X)) = \phi(n,d,k)$, the number defined above. Remarkably, however, we will be left neither with a way to determine whether a given hypersurface has the expected-dimensional Fano variety, nor with a sure-fire means of producing examples in any particular case—even through in the complex case we could randomly choose such a hypersurface with probability 1 (if such a thing were possible).

1.1. **Definition of the Fano variety.** Let V be an $(n+1)$-dimensional vector space over an algebraically closed field K, and let $X \subset \mathbb{P}(V) \cong \mathbb{P}^n$ be a projective variety. Define the k'th *Fano Variety* $F_k(X)$ as the set of projective k-planes contained in X,

$$F_k(X) = \{\Lambda \in \mathbb{G}(k,n) \mid \Lambda \subset X\}.$$

In this section we will show that $F_k(X)$ is in fact a projective subvariety of the Grassmannian, once we have recalled some basic facts.

We will view $\mathbb{G}(k,n)$ as a closed subvariety of $\mathbb{P}(\bigwedge^{k+1} V)$ via the Plucker embedding

$$\Lambda = \langle v_0, \dots, v_k \rangle \mapsto [v_0 \wedge \cdots \wedge v_k].$$

We will also occasionally refer to the Grassmannian $G(k+1,n+1)$ of $(k+1)$-dimensional linear subspaces of V, and of course $G(k+1,n+1) = \mathbb{G}(k,n)$. We will sometimes for emphasis write $G(k+1,V)$ specifically for the Grassmannian of $(k+1)$-planes in V, and likewise $\mathbb{G}(k,\mathbb{P}V) = G(k+1,V)$.

Proposition 1.1. *The variety $\mathbb{G}(k,n)$ is covered by open sets isomorphic to $\mathbb{A}^{(k+1)(n-k)}$. Therefore,*

$$\dim(\mathbb{G}(k,n)) = (k+1)(n-k).$$

Proof. Decompose V as a direct sum $V = V_0 \oplus W_0$, with $\dim(V_0) = k+1$ and $\dim(W_0) = n-k$. Choose a basis $\{v_0, \dots, v_k, w_{k+1}, \dots, w_n\}$ for this direct sum, and take coordinates on V dual to this basis. Let $U \subset G(k+1,n+1)$ be the set of $(k+1)$-dimensional linear subspaces of V complimentary to W_0 (which includes V_0). Then U is the intersection with $G(k+1,n+1)$ of the affine patch of the ambient projective space $\mathbb{P}(\bigwedge^{k+1} V)$ consisting of points whose $(v_0 \wedge v_1 \wedge \cdots \wedge v_k)$-coordinate is nonzero. Hence U is open and affine.

For a $(k+1)$-plane $\Lambda \in U$, define the vectors $\Lambda_i = (v_i + W_0) \cap \Lambda$, for $i = 0 \dots k$. Then $\{\Lambda_i\}$ is a basis for Λ, and we can represent Λ by the $(k+1) \times (n-k)$ matrix M_Λ with rows Λ_i:

$$M_\Lambda = \begin{pmatrix} 1 & 0 & \cdots & 0 & a_{11} & \cdots & a_{1(n-k)} \\ 0 & 1 & \cdots & 0 & a_{21} & \cdots & a_{2(n-k)} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & a_{(k+1)1} & \cdots & a_{(k+1)(n-k)} \end{pmatrix}. \tag{2}$$

The Plucker coordinates on U are the $(k+1) \times (k+1)$ minors of this matrix. One can see that each entry a_{ij} of the $(k+1) \times (n-k)$ submatrix (a_{ij}) is in fact the value of some one of these minors. So, the a_{ij} are in fact affine coordinates on $U \cong \mathbb{A}^{(k+1)(n-k)}$. Each affine coordinate patch (i. e. complement of a *coordinate* hyperplane) of the ambient space meets $\mathbb{G}(k,n)$ in a set U of this form, which implies that $\dim(\mathbb{G}(k,n)) = (k+1)(n-k)$. ☐

Thus in order to show that $F_k(X)$ is a closed subvariety of $\mathbb{G}(k,n)$, it is sufficient to show that $F_k(X) \cap U$ is an affine variety, for any such affine patch $U \subset \mathbb{G}(k,n)$. Wlog, let X be a hypersurface defined by a single homogeneous polynomial $f(X_0,\ldots,X_n)$ of degree d; the general case will follow just by taking intersections. For X a hypersurface, $\Lambda \subset X$ iff the restriction of f to Λ vanishes identically. Let $\Lambda \in U$, and parametrize the linear space Λ by $[u_0,\ldots,u_k] \mapsto [\sum u_i\Lambda_i]$, with Λ_i the i'th row of the matrix M_Λ from (2) (i. e. basis vector for Λ).

We then simply plug in this parametrization to obtain a polynomial $f(\sum u_i\Lambda_i)$ of the same homogeneous degree d in the parameters $\{u_i\}$ of the space Λ. The coefficients are then inhomogeneous polynomials in the a_{ij}; their vanishing is necessary and sufficient for f to vanish identically on Λ, i. e. for the space Λ to be contained in the locus $\{f=0\}=X$. Thus we have obtained polynomial equations in the a_{ij} defining $\{\Lambda \in U \mid \Lambda \subset X\} = F_k(X) \cap U$, for any such U. Hence $F_k(X)$ is a closed subvariety of $\mathbb{G}(k,n)$, and a projective variety itself.

Note that if X is quasi-projective, with projective closure $\overline{X}$, then the Fano Variety $F_k(X) \subset F_k(\overline{X})$ is a locally closed subvariety. This is simply because for any two families of subvarieties of a variety, the subset of disjoint pairs is open ([13], Ch. 4).

Remark 1.2. Elimination theory can be used on each element of a finite cover of patches $\{U_i \cong \mathbb{A}^{(k+1)(n-k)}\}$, using the explicit polynomial description of $F_k(X) \cap U_i$ just given, to answer the opening question of whether or not a given variety X actually contains a k-plane.

1.2. Estimating the dimension of $F_k(X)$.

1.2.1. *Incidence correspondence.* In order to find an estimate of the dimension of $F_k(X)$ for a hypersurface, we set up an incidence correspondence between hypersurfaces and k-planes. Let $\mathbb{P}^N$ be the projective space of homogeneous polynomials of degree d in $n+1$ variables, so $N = \binom{n+d}{d} - 1$. Then let

$$I = \{(f,\Lambda) \in \mathbb{P}^N \times \mathbb{G}(k,n) \mid f(\Lambda) = 0\}.$$

Given any two families of projective varieties in $\mathbb{P}^n$, the set of pairs such that the first subvariety is contained in the second is a constructible set (see [13]). Here, I is in fact a closed subvariety of $\mathbb{P}^N \times \mathbb{G}(k,n)$, which can be shown along much the same lines as the above argument for $F_k(X)$. Choose an affine coordinate patch U_0 of $\mathbb{P}^N \times \mathbb{G}(k,n)$. Under the Segre embedding, $U_0 = U_1 \times U_2$ for two affine coordinate patches $U_1 \subset \mathbb{P}^N$ and $U_2 \subset \mathbb{G}(k,n)$. Then U_1 is of the form

$$U_1 = \{[f(T)] \in \mathbb{P}^N \mid f(T) = T^{\underline{h_0}} + \sum_{\underline{h} \neq \underline{h_0}} e_{\underline{h}} T^{\underline{h}}\},$$

and we let $U_2 \subset \mathbb{G}(k,n)$ have affine coordinates a_{ij} coming from the matrix M_Λ of (2). Then via the Segre embedding, one obtains inhomogeneous coordinates on $U_0 = U_1 \times U_2$ as pairs $(\{e_{\underline{h}}\},\{a_{ij}\})$.

Meanwhile, the points $f \in U_1$ are still homogeneous polynomials, and the points $\Lambda \in U_2$ are still projective linear spaces. So we can parametrize Λ as before by $[u_0,\ldots,u_n] \mapsto [\sum u_\ell \Lambda_\ell]$. Then $f(\Lambda)$ is a homogeneous polynomial in the parameters u_ℓ, whose coefficients are doubly inhomogeneous polynomials in the coefficients $e_{\underline{h}}$ of f and the entries a_{ij} of the matrix M_Λ. These coefficients vanish exactly when $f(\Lambda) \equiv 0$, so they define I as a variety on the affine coordinate patch U_0. Hence I is a projective subvariety of $\mathbb{P}^N \times \mathbb{G}(k, n)$, as desired.

1.2.2. *Dimension estimate.* Now we can use I to estimate the dimension of $F_k(X)$ for a hypersurface X. Consider the restriction to I of the projections onto the factors of $\mathbb{P}^N \times \mathbb{G}(k, n)$,

$$\begin{array}{ccc} & I & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathbb{P}^N & & \mathbb{G}(k,n). \end{array}$$

The fiber of π_1 over a point $X \in \mathbb{P}^N$ is the Fano variety $F_k(X)$. So we hope to estimate this dimension using the following fundamental result. Recall that a function $f : X \to \mathbb{R}$ is upper-semicontinuous if $f^{-1}([r, \infty))$ is closed for all $r \in \mathbb{R}$. If furthermore $f : X \to \mathbb{Z}$ takes integer values, then f attains its minimum on an open set of X.

Proposition 1.3. (Theorem on fiber dimension) *Let X be quasi-projective and $\varphi : X \to \mathbb{P}^n$ a regular map, and let $Y = \overline{\varphi(X)}$ be the closure of the image. For each $p \in X$, let $X_p = \varphi^{-1}(\varphi(p))$ be the fiber of φ through p. Then,*
(a) *For all $p \in X$,*

$$\dim_p(X_p) \geq \dim_p(X) - \dim_{\varphi(p)}(Y),$$

with equality on a nonempty open subset of X. The function $\dim_p(X_p)$ is upper-semicontinuous on X, i. e. for any integer k the set of p such that $\dim_p(X_p) \geq k$ is closed.

(b) *If X is projective, then for all $q \in Y$,*

$$\dim(\varphi^{-1}(q)) \geq \dim(X) - \dim(Y),$$

with equality on a nonempty open subset of Y. The function $\dim(\varphi^{-1}(q))$ is an upper-semicontinuous function of $q \in Y$.

Proof. This is Theorem 11.12 and Cor. 11.13 of [13], and Section I.6.3, Theorem 7 of [15]. □

Corollary 1.4. (a) *Let X_0 be an irreducible component of X, $Y_0 = \overline{\varphi(X_0)}$, and $\mu = \min_{p \in X_0}(\dim_p(X_p))$. Then $dim(X_0) = dim(Y_0) + \mu$.*

(b) *For X projective, let X_0 be an irreducible component and let $\lambda = \min_{q \in Y}(\dim(\varphi^{-1}(q)))$. Then $dim(X_0) = dim(Y_0) + \lambda$.*

Proof. These are attached to the theorem in [13] and [15]. □

Corollary 1.5. *Let X be an irreducible quasi-projective variety, and $\varphi : X \to Y$ be a dominant morphism. If any fiber of φ contains an isolated point, then $\dim(X) = \dim(Y)$ and the general fiber of φ has dimension zero.*

Proof. Let $p \in X_p$ be an isolated point. Then

$$\dim_p(X_p) = 0 = \dim(X) - \dim(\overline{\varphi(X)}) = \dim(X) - \dim(Y),$$

by Corollary 1.4(a). The second claim is immediate from 1.3(a). □

The results of this chapter will come from applying these statements to the map π_1 as well as in several similar instances. First, we apply Proposition 1.3 to the map π_2 to compute the dimension of I. Given $\Lambda \in \mathbb{G}(k,n)$ consider the surjective linear map

$$\{\text{polynomials of degree } d \text{ on } \mathbb{P}^n\} \longrightarrow \{\text{polynomials of degree } d \text{ on } \Lambda \cong \mathbb{P}^k\}$$

given by restriction to Λ. The kernel of this map is a linear subspace of dimension $\binom{n+d}{d} - \binom{k+d}{d}$, whose projectivization is the the fiber of π_2 over Λ. Therefore the fibers of π_2 are all irreducible of the same dimension, so I is irreducible (by Theorem 11.14 of [13]). From Proposition 1.3,

$$\dim(I) = \dim(\mathbb{G}(k,n)) + \dim(\pi_2^{-1}(\Lambda)) = (k+1)(n-k) + \binom{n+d}{d} - \binom{k+d}{d} - 1.$$

The map π_1 cannot be dealt with so straightforwardly. However, it will be shown in section 1.4 that its image has maximal dimension (unless $d = 2$). This means that π_1 is injective when $\dim(I) \leq N$, and π_1 is surjective when $\dim(I) \geq N$. Once we establish this, we will be able to count the dimension by applying Proposition 1.3. Until this last fact was proven, by [1] in 1987, the following was merely a "dimension estimate:"

Theorem 1.6. *For $n \geq k \geq 0$ and $d \geq 3$, let*

$$\phi(n,d,k) = \dim(I) - \dim(\mathbb{P}^N) = (k+1)(n-k) - \binom{k+d}{d}.$$

(a) *For $\phi(n,d,k) < 0$, the subvariety of $\mathbb{P}^N$ of hypersurfaces that contain a k-plane has codimension $-\phi$.*
(b) *For $\phi(n,d,k) = 0$, every hypersurface of degree d in $\mathbb{P}^n$ contains a k-plane, and a general hypersurface contains a positive number of k-planes.*
(c) *For $\phi(n,d,k) > 0$, a hypersurface X has $\dim(F_k(X)) \geq \phi(n,d,k)$, with equality for general X.*

By the preceding discussion, this theorem will be proved if we can establish that the image of $\pi_1 : I \to \mathbb{P}^N$ has maximal dimension, i. e. is surjective for $\phi \geq 0$ and injective for $\phi \leq 0$. At present, however, we can prove only small pieces of this theorem. It is clear that for $\phi < 0$, the variety of hypersurfaces that do contain a k-plane has codimension at least $-\phi = \dim(\mathbb{P}^N) - \dim(I)$, so the following is already evident:

Corollary 1.7. *For $\phi < 0$, a general hypersurface contains no k-planes.*

Also note that for $\phi > 0$, if a hypersurface X contains a k-plane (i. e. lies in the image $\pi_1(I)$) then it contains infinitely many k-planes, since $\dim(F_k(X)) = \dim(\pi_1^{-1}(X)) \geq \phi(n,d,k) > 0$. The nontrivial task is to show that a hypersurface does in fact contain a k-plane:

Example 1.8. The general quadric threefold in $\mathbb{P}^4$ contains no 2-planes—this is the case $n = 4, k = 2, d = 2$ and $\phi(4,2,2) = 0$. For, in the family of all quadric hypersurfaces, the sub-family of cones over quadric surfaces has codimension 1; and each such cone contains at least a 1-dimensional family of 2-planes. Therefore this sub-family must equal the image $\pi_1(I)$. The stipulation $d \geq 3$ is thus necessary in (b), and as such it is not obvious that (b) holds at all.

Meanwhile, in case (c) there is a trivial partial solution to this problem: for $n \geq \binom{d+k-1}{d-1} + k - 1$, any hypersurface $X \subset \mathbb{P}^n$ is *swept out* by k-planes (by Lemma 2.11 to follow). This prevents such a failure as occurs for quadric threefolds in Example 1.8: the map $\pi_1 : I \to \mathbb{P}^N$ is surjective, therefore $\dim(F_k(X)) \geq \dim(I) - \dim(\mathbb{P}^N) = \phi(n,d,k)$ with equality for general X. This suggests the fruitfulness of the low-degree limit, to which we will turn in subsequent chapters. But below this point, the existence of even a single k-plane is nontrivial.

The method of proof will be as follows. In order to establish parts (a) and (b) of Theorem 1.6, we will use Corollary 1.5: we must simply demonstrate the existence of a hypersurface whose Fano scheme has an isolated point. The fact that such hypersurfaces exist in general will be shown fully in Section 1.4. And similarly, Proposition 1.3 will imply (c) if we can prove the existence of a hypersurface X containing a k-plane Λ such that $\dim_\Lambda(F_k(X)) = \phi$, i. e. that $F_k(X)$ has the expected dimension locally at Λ. In section 1.4, we will show that the existence of such an X is implied by the full algebraic result of [1]. In summary,

Remark 1.9. *To establish Theorem 1.6 for $\phi \leq 0$ (parts (a) and (b)), it is sufficient to show the existence of a hypersurface X_0 whose Fano variety contains an isolated point. To establish the theorem for $\phi \geq 0$ (part (c)), we need* $\dim(F_k(X_0)) = \phi$.

1.3. **Proofs by example.** Here we give some "proofs by example" of Theorem 1.3(b) for particular choices of n, d, k for which $\phi(n,d,k) = 0$, i. e. we exhibit hypersurfaces whose Fano variety has an isolated point. By Corollary 1.5 to the Theorem on Fiber Dimension, these are, in fact, proofs that all degree d hypersurfaces in $\mathbb{P}^n$ contain k-planes.

1.3.1. *The case $k = 1$.* We first attend to the case of lines.

Example 1.10. Let $k = 1$, and let n and d be integers such that

$$\phi(n,d,1) = 2n - 3 - d = 0.$$

One can readily exhibit a degree d hypersurface in $\mathbb{P}^n$ with an isolated line. Note that $(d-1)/2 = n-2$, and take coordinates $[Z_0, Z_1, W_0, \ldots W_{n-2}]$ on $\mathbb{P}^n$. Consider the hypersurface X_0 defined by the polynomial

$$\begin{aligned} F(Z,W) = &\quad \textstyle\sum_{k=0}^{(d-1)/2} W_k Z_0^{d-1-k} Z_1^k \\ = &\quad W_0 Z_0^{d-1} + W_0 Z_0^2 Z_1^{d-1-2} \quad + \cdots + W_{n-2} Z_1^{d-1}. \end{aligned}$$

Then $\ell_0 = \{W_i = 0\}$ is an isolated line of $F_k(X_0)$. For, taking coordinates a_{ij} ($i = 0, 1$ and $j = 1, \ldots n-1$) on the affine space U of lines ℓ complementary to the $(n-2)$-plane $\{Z_0 = Z_1 = 0\}$, we get

$$F(\ell) = (a_{01}Z_0 + a_{11}Z_1)Z_0^{d-1} + (a_{02}Z_0 + a_{12}Z_1)Z_0^{d-1-2}Z_1^2 + \cdots + (a_{0(n-1)}Z_0 + a_{1(n-1)}Z_1)Z_1^{d-1}.$$

Then clearly $\ell_0 = (a_{ij} = 0)$ is the only line in U such that $F(\ell_0) = 0$, hence ℓ_0 is an isolated point of the Fano variety. As one case, we have proved the following:

Corollary 1.11. *A cubic hypersurface in $\mathbb{P}^3$ contains lines.*

Moreover, it is possible to obtain parts (a) and (c) of Theorem 1.6 in the case $k = 1$ by appropriately truncating or adding terms to the polynomial $F(Z,W)$.

1.3.2. *The case $k = 2$ and above.* We give similar examples in the cases $k = 2$, $d = 4, 5$ and $\phi = 0$. From these examples we can inductively describe polynomials as required for all $d \equiv 1, 2 \mod 3$, proving all cases $k = 2$ and $\phi = 0$.

Example 1.12. [4] **(a)** $(n = 7, d = 4, k = 2)$ The quartic hypersurface in

$$\mathbb{P}^7 = \{[Z_0, Z_1, Z_2, W_0, W_1, W_2, W_3, W_4]\}$$

defined by the equation

$$W_0 Z_0^3 + W_1 Z_1^3 + W_2 Z_2^3 + W_3 Z_0 Z_1 Z_2 + W_4 (Z_0 Z_1^2 + Z_1 Z_2^2 + Z_2 Z_0^2) = 0,$$

contains the 2-plane $\{W_i = 0\}$ as an isolated point of its Fano variety of 2-planes.

(b) $(n = 9, d = 5, k = 2)$ The quintic hypersurface in $\mathbb{P}^9 = \{[Z_0, Z_1, Z_2, W_0, \ldots, W_6]\}$ defined by the equation

$$\sum_{i,j,k \text{ even}} W_\ell Z_0^i Z_1^j Z_2^k + W_6 (Z_0^2 Z_1 Z_2 + Z_0 Z_1^2 Z_2 + Z_0 Z_1 Z_2^2) = 0$$

contains the 2-plane $\{W_i = 0\}$ as an isolated point of its Fano variety of 2-planes.

(c) $(n = 8, d = 3, k = 3)$ The cubic hypersurface in $\mathbb{P}^8 = \{[Z_0, \ldots, Z_3, W_0, \ldots, W_4]\}$ defined by the equation

$$\sum_{i=0}^{3} W_i Z_i^2 + W_4 (Z_0 Z_1 + Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_0) = 0$$

contains the 3-plane $\{W_i = 0\}$ as an isolated point of its Fano variety of 3-planes.

To prove these assertions, one must show that upon substituting $W_i = \sum a_{ij} Z_j$ the resulting forms span $K[Z]_d$ as a_{ij} varies. Then, since $\phi(n, d, k) = 0$, they are also linearly independent, meaning that $\Lambda = (a_{ij} = 0 \; \forall \; i, j)$ is the only k-plane contained in the affine patch $U \cap F_k(X)$. This is seen in part (a) by choosing values of the coefficients a_{ij} as follows: setting $W_i = Z_i$ for each $i = 0, 1, 2$ individually, we see that all of the monomials Z_i^4 are included in the span. Setting $W_3 = Z_i$ for each i individually, we get each monomial involving all of the Z_k. Now, setting $W_4 = Z_i$, the only monomial not divisible by any Z_k^3 or involving all three Z_i is $Z_i^2 Z_{i+1}^2$, and these are exactly the remaining monomials. So we have shown that all the monomials of $K[Z]_4$ are in the span. Parts (b) and (c) are treated similarly.

For the case $k = 2$, one can describe inductively a set of polynomials defining hypersurfaces with the required property. We have treated the cases $d = 4, 5$ individually in Example 1.12(a), (b), respectively. Then, given $F_d(Z, W)$ of degree d as required, define

$$F_{d+3}(Z, W, W') = Z_0 Z_1 Z_2 F_d(Z, W) + W'_a Z_0^{d+2} + W'_b Z_1^{d+2} + W'_c Z_2^{d+2}$$
$$+ \sum_{k=1}^{d} W'_k (Z_0^{k+1} Z_1^{d+1-k} + Z_1^{k+1} Z_2^{d+1-k} + Z_2^{k+1} Z_0^{d+1-k}).$$

[4]Part (a) here was found by writing out the the result of substituting $W_i = \sum a_{ij} Z_j$ into an arbitrary polynomial, and choosing its coefficients so that the result spans $K[Z]_4$. Part (b) was found by writing out the degree d monomials in a 2-simplex with the degree $d - 1$ monomials interlaced, and choosing a polynomial that spans $K[Z]_5$ under certain choices of coefficients a_{ij}. Part (c) was found similarly.

This will again define a hypersurface having $\{W_i = W'_{i'} = 0\}$ as an isolated 2-plane. This proves Theorem 1.6(b) in the case of 2-planes.

In the case $d = 3$, $k = 2$, it is possible to find a polynomial whose Fano variety has local dimension 2. Then the same formula can be used to extend to all $d \equiv 0 \bmod 3$. It is not clear how to generalize either the base cases or the induction method to $k > 2$.

1.4. $F_k(X)$ **has the estimated dimension for general** X**.** In this section we prove Theorem 1.6 (a) and (b) in general, and show that (c) is implied by the main theorem of [1].

1.4.1. *An algebraic lemma.* The following lemma, a partial version of the result of [1], will show directly the existence of the polynomials we require. We adopt notation similar to that of [1].

Let $K[x]$ be the ring of polynomials in r variables $x_1, \dots, x_r$. Let

$$N(r, d) = \dim_K(K[x]_d) = \binom{d+r-1}{d}$$

be the dimension of the homogeneous part of degree d (unrelated to the number $N(d, k)$ defined later in Section 2.5).

In the proof of the lemma we will use several other spaces from [1]. Let P be the projective space of $m \times r$ matrices with entries in K. Let $B = (K[x]_{d-1})^m$ be the affine space of m-tuples of degree $d-1$ forms in r variables. Let $\vec{x} = (x_1, \dots, x_r)^t$ be the column vector with the formal variables as entries. If $[A] \in P$ is represented by the matrix A, then $bA\vec{x}$ is a 1×1 matrix whose unique entry is a polynomial expression of homogeneous degree d in the variables x_i.

In the lemma, we will use a strategy surprisingly similar to the central proof of the chapter. Let

$$V = \{(b, [A]) \in B \times P \mid bA\vec{x} = 0\},$$

a closed subvariety of $B \times P$. Then a form $b = (F_1, \dots F_m)$ is in the image of the projection $\pi_1(V)$ onto the first factor if and only if there exists a nontrivial linear dependence between the mr forms $\{x_j F_i\}$, i. e. $bA\vec{x} = 0$ for some nonzero matrix $A \in K^{m \times r}$. We will use V to bound the dimension of the subvariety of B consisting of m-tuples that result in such a linear dependence.

Finally, let $P_t \subset P$ be the irreducible quasi-projective subvariety of P consisting of points representable by a matrix of rank exactly t (see [13]), and let $V_t = V \cap (B \times P_t)$.

Lemma 1.13. *Let $d \geq 3$ and $m \geq r$ be positive integers such that $N(r, d) = mr + \ell$ for some nonnegative integer ℓ. Let F_i, $i = 1, \dots m$, be generic forms in $K[x]_{d-1}$. Then the mr forms $\{x_j F_i\}$, for $i = 1, \dots, m$ and $j = 1, \dots r$, are linearly independent in $K[x]_d$.*

Proof. This claim comes from the main idea of [1] by a much shorter argument, due to the restriction $\ell \geq 0$ (which amounts to $\phi \leq 0$).

Let π_2 be the restriction of the projection onto the second factor $V \hookrightarrow B \times P \to P$. It certainly may be possible that V is reducible, which makes the situation here nontrivial.

Let $0 \leq t \leq r$, so π_2 restricts to a map $V_t \to P_t$. We will apply the Theorem on Fiber Dimension to these maps to eventually determine that

$$\dim(V) = mN(r, d-1) - 1, \tag{3}$$

i. e. $\dim(V) = \dim(B) - 1$, and a generic m-tuple $(F_i) \subset B$ will satisfy the conclusion of the lemma.

Let $p = [(a_{ij})] \in P_t$ be represented by the rank t matrix (a_{ij}), and let $F = \pi_2^{-1}(p)$ be the fiber of π_2 over p. We will first show that this fiber is irreducible and has dimension

$$\dim(F) = mN(r, d-1) - N(r, d) + N(r-t, d). \tag{4}$$

Since its unique P-coordinate is p, the variety F may be identified with its projection onto B, i. e. the set of vectors $b = (F_1, \ldots, F_m) \in (K[x]_{d-1})^m$ such that $b(a_{ij})\vec{x} = 0$. So letting L be the column vector of linear forms $(a_{ij})\vec{x} = L = (L_1, \ldots, L_m)^t$, we think of F as the affine cone $\{b \in B \mid b \cdot L = 0\}$. So F is simply a linear subspace of the vector space B, hence irreducible.

Let $J \subset k[x]$ be the ideal generated in $L_1, \ldots, L_m$. Then we have the short exact sequence of graded rings

$$(K[x](-1))^m \xrightarrow{\cdot L} K[x] \to K[x]/J.$$

Since $F = \text{Ker}(\cdot L : (K[x](-1))^m \to K[x])$, this induces the short exact sequence of vector spaces

$$0 \to F \to (K[x]_{d-1})^m \xrightarrow{L} K[x]_d \to (K[x]/J)_d \to 0.$$

Since $K[x]/J$ is isomorphic to a polynomial ring in $r-t$ variables, this sequence directly yields the desired dimension count for $\dim(F)$ in (4).

Now, notice that the map π_2 is surjective onto P, since $(0, p) \in V \, \forall p \in P$. Furthermore, $\dim(P_t) = mr - 1 - (m-t)(r-t) = N(r, d) - \ell - 1 - (m-t)(r-t)$ by [13] Prop. 12.2 and the given. So, since the fiber F was arbitrary, from Corollary 1.4 to the Theorem on Fiber Dimension we have

$$\begin{aligned} \dim(V_t) &= \dim(F) + \dim(P_t) \\ &= mN(r, d-1) - N(r, d) + N(r-t, d) \\ &\quad + N(r, d) - \ell - 1 - (m-t)(r-t) \\ &= (mN(r, d-1) - 1) + N(r-t, d) - (m-t)(r-t) - \ell. \end{aligned}$$

To show 3, we now must prove that $(m-t)(r-t) + \ell \geq N(r-t, d)$ for all $t \leq r$, with our choice of m and r. Proceed by induction on t: the base case $t = 0$ follows from the assumption $N(r, d) = mr + \ell$. Now assume $(m-(t-1))(r-(t-1)) + \ell \geq N(r-(t-1), d)$:

$$\begin{aligned} (m-t)(r-t) &= \frac{r-t}{r-t+d}(m-t)(r-t+d) \\ &\geq \frac{r-t}{r-t+d}(m-t+1)(r-t+1) \text{ since } m \geq r \geq t \text{ and } d-1 \geq 2 \\ &\geq \frac{r-t}{r-t+d}(N(r-t+1, d) - \ell) \text{ by induction} \\ &\geq N(r-t, d) - \ell. \end{aligned}$$

We have established that for all $t \leq r$, $\dim(V_t) \leq \dim(V_r) = mN(r, d-1) - 1$. Since $V = \bigcup V_t$, we get $\dim(V) = \max_t(\dim(V_t)) = \dim(V_r) = mN(r, d-1) - 1$, as desired.

We have shown $mN(r,d-1)-1 = \dim(V) < mN(r,d-1) = \dim(B)$, so $\pi_1(V)$ must be a proper subvariety of $B = (K[x]_{d-1})^n$. Thus an m-tuple $(F_1,\dots,F_n) \in B - \pi_1(V)$, i. e. a generic n-tuple, yields a linearly independent set $\{x_j F_i\}$ in $K[x]_d$. □

1.4.2. *Proof of Theorem 1.6.* Recall that as per Remark 1.9, in order to prove Theorem 1.6 we must simply show the existence of certain hypersurfaces.

Let d and $n > k$ positive integers, and take coordinates $[Z_0,\dots,Z_k,W_1,\dots,W_{n-k}]$ on $\mathbb{P}^n$. We will restrict our attention to a hypersurface X_0 defined by a polynomial of the form

$$F(Z,W) = \sum_{i=1}^{n-k} W_i F_i(Z_0,\dots,Z_k),$$

where each F_i is a homogeneous polynomial of degree $d-1$. The hypersurface X_0 then contains the k-plane $\Lambda_0 = \{W_1 = W_2 = \cdots = W_{n-k} = 0\}$.

Consider the affine patch U of the Grassmannian consisting of k-planes complementary to the $(n-k-1)$-plane $\{Z_0 = \cdots = Z_k = 0\}$. Here we have coordinates a_{ij} from (2), with Λ_0 corresponding to $a_{ij} = 0$. A plane $\Lambda \in U$ is parametrized as

$$[Z_0,\dots,Z_k] \mapsto \left[Z_0,\dots,Z_k,\sum a_{1j}Z_j,\dots,\sum a_{(n-k)j}Z_j\right].$$

The restriction $F|_\Lambda$ is then given simply by

$$F|_\Lambda(Z_0,\dots,Z_k) = \sum_{i=1}^{n-k}(\sum_j a_{ij}Z_j)F_i(Z_0,\dots,Z_k),$$

and a hyperplane Λ lies on X_0 if and only if this restriction vanishes identically.

Now, assume that the F_i are generic forms in $K[Z]_{d-1}$, and $\phi(n,d,k) = (k+1)(n-k) - \binom{k+d}{d} \leq 0$. The conditions of Lemma 1.13 are met if we take $m = n-k$, $r = k+1$, for then

$$N(r,d) = \binom{r+d-1}{d} = \binom{k+d}{d} \geq (k+1)(n-k) = rm.$$

So from the lemma we may conclude that the $(k+1)(n-k)$ forms $\{Z_j F_i(\underline{Z})\}$ are linearly independent in $K[\underline{Z}]$. But this is exactly the statement

$$F|_\Lambda(Z) = \sum_{i=1}^{n-k}(\sum_j a_{ij}Z_j)F_i(Z_0,\dots,Z_k) \equiv 0 \text{ iff } a_{ij} = 0 \text{ for all } i,j.$$

Thus $\Lambda_0 = (a_{ij} = 0\,\forall\, i,j)$ is the unique point of $F_k(X) \cap U$, hence an isolated point, as desired. This establishes parts (a) and (b) of Theorem 1.6.

The full result of [1] adds to Lemma 1.13 the fact that for $N(r,d) > rm$, the linear map

$$\begin{array}{rcl} K^{rm} \cong K[Z]_1 \otimes K[Z]_{d-1} & \to & K[Z]_d \cong K^{N(r,d)} \\ Z_j \otimes F_i(Z) & \mapsto & Z_j F_i(Z) \end{array}$$

is surjective, for $d \geq 3$.[5] Hence the kernel of this map has dimension $rm - N(r, d) = \phi(n, d, k)$ in our set-up, which is exactly the statement that $F|_\Lambda(Z) \equiv 0$ for a_{ij} in a linear subspace of dimension $\phi(n, d, k)$. Thus $F_k(X_0) \cap U \subset U \cong \mathbb{A}^{rm}$ is a linear subspace of dimension ϕ, which is therefore the local dimension of $F_k(X_0)$ on U. This establishes part (c) of the theorem. □

Remark 1.14. It is somewhat surprising that our argument establishes surjectivity only in the case $\phi(n, d, k) = 0$, and not for the case that the dimension of I is strictly bigger than that of the target $\mathbb{P}^N$. However, this is not the only failure of its kind, e. g. a surjective map of sheaves is not in general surjective on global sections unless it is also injective.

[5]The requirement $d \geq 3$ is not included in [1]. The error is simply the unjustified claim that the inequality at the bottom of p. 237 holds for "$d = 1$" (which is $d - 1 = 1$ in this paper). The subsequent combinatorial proof of the inequality for $d \geq 2$ is correct.

2. Fano Varieties in the Low-Degree Limit.

The goal of this chapter will be to specify when a hypersurfaces is "general" in the sense of the previous chapter, meaning that its Fano variety of k-planes has the expected dimension ϕ (the number defined in Theorem 1.6). Our answer will be that provided $d \ll n$, this holds for all *smooth* hypersurfaces of degree d in $\mathbb{P}^n$. Our main reference for this chapter and the next is the 1998 paper [2] by Harris, Mazur, and Pandharipande.

2.1. **Notation and terminology.** Throughout the rest of the paper, we work over a fixed algebraically-closed field K of characteristic 0. By a *scheme*, we will mean a quasi-projective K-scheme, i. e. a locally closed subscheme of $\mathbb{P}^n$. A reduced scheme will then be called a *variety* (not necessarily irreducible), which is therefore just a classical quasi-projective variety. So for our purposes, an integral K-scheme is an irreducible quasi-projective variety. Furthermore, we will have occasion to consider only closed (i. e. geometric) points of schemes: by a "point" $x \in X$ we will mean a closed point.

If W and T are B-schemes, we will abbreviate $W_T = W \times_B T$ for their fiber product.

Definition 2.1. A reduced B-scheme $P \to B$ will be called a *projective bundle* of rank r, or a "$\mathbb{P}^r$-bundle," if for each $b \in B$ there exists a Zariski-open neighborhood $U \ni b$ yielding a commutative diagram

$$\begin{array}{ccc} P_U & \xrightarrow{\sim} & U \times \mathbb{P}^r \\ & \searrow & \downarrow \\ & & U. \end{array}$$

Remark 2.2. *(a)* In this paper, all projective bundles will be sub-bundles of some trivial bundle $B \times \mathbb{P}^n = B \times \mathbb{P}V$. For our purposes, the operation"$\mathbb{P}$" will send a vector sub-bundle of $B \times V$ to the corresponding projective sub-bundle of $B \times \mathbb{P}V$ (which is simply $\mathrm{Grass}_0(E)$ as defined below). Thus every projective bundle P will be equal to $\mathbb{P}(E)$ for a vector sub-bundle $E \subset B \times V$. Where convenient, we will thus be able to work equivalently in the context of vector bundles.

(b) There frequently arise bundles $X \to B$ such that each fiber X_b is isomorphic to a projective space $\mathbb{P}^r$, while the bundle fails to be locally trivial in the Zariski topology: an example is the family of smooth conics in the plane, which does not even admit a rational section (see [13], ch. 4). In fact, this failure can be construed as the main difficulty encountered in Chapter 3.

However, if the given projective bundle is a sub-bundle of a trivial bundle, i. e. $X \subset B \times \mathbb{P}^n \to B$, and the scheme-theoretic fibers are reduced linear subspaces $\mathbb{P}^r \subset \mathbb{P}^n$, then $X \to B$ is in fact locally trivial: this is Proposition 3.3, to follow.

2.2. **A first result.** To give a first demonstration of the techniques of the remainder of the paper (residual varieties in particular) we will prove a well-known result. The following is actually the simplest case of the main result of the next chapter, Theorem 3.6; but the techniques are quite similar to those used in the proof of the main result of this chapter, Theorem 2.13—which will in turn be necessary for the proof of Theorem 3.6. The proof crucially involves the question of whether a bundle with projective fibers is actually a "projective bundle" as per Definition 2.1.

Proposition 2.3. *For $n \geq 3$, a smooth cubic hypersurface $X \subset \mathbb{P}^n_K$ is unirational, provided K is algebraically closed and char$(K) = 0$.*

Proof. Certain notions used here (such as Grassmann bundles) will be made rigorous in Section 2.3 immediately following this proof, which will thus be somewhat casual.

From Theorem 1.6, we may choose a plane Γ of some dimension $l > 0$ lying on X (a line, if one prefers). Then the variety of $(l+1)$-planes containing the l-plane Γ is parametrized by $\mathbb{P}^{n-l-1} = \mathbb{P}^n/\Gamma$. Since X is smooth, a general such $(l+1)$-plane $\Theta \supset \Gamma$ meets X in a the union of the plane Γ and a "residual" quadric hypersurface of dimension l, call it Q_Θ. So, X is birational to the total space of a "quadric bundle"—i. e. a $\mathbb{P}^{n-l-1}$-scheme whose fibers are quadrics—over an open subset of $\mathbb{P}^{n-l-1}$.

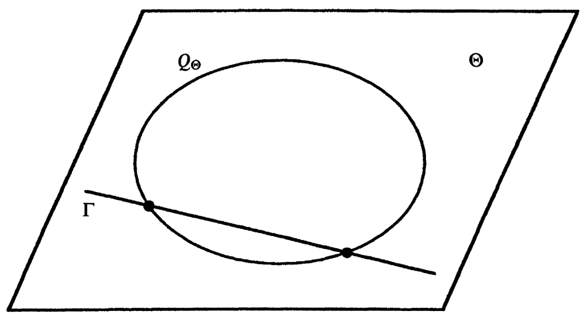


FIGURE 1

To be precise, the quadrics Q_Θ are the generic fibers of the blow-up

$$\pi_\Gamma : \tilde{X} = \mathrm{Bl}_\Gamma(X) \longrightarrow \mathbb{P}^{n-l-1},$$

which we view as a quadric bundle over $\mathbb{P}^{n-l-1}$ via the projection π_Γ from $\Gamma \cong \mathbb{P}^{n-l-1}$. The proper transform $\mathrm{Bl}_\Gamma(\Theta)$ of an $(l+1)$-plane $\Theta \supset \Gamma$ is again an $(l+1)$-plane, and $Q_\Theta = \mathrm{Bl}_\Gamma(\Theta) \cap \tilde{X}$. (In the rest of this argument we will sometimes fail to mention when a property holds only over an open, dense subset.)

We will show that $\tilde{X}$ is unirational by viewing it as a quadric bundle. Each irreducible quadric Q_Θ is rational individually: projecting from any point of Q_Θ to a hyperplane of Θ is 1-1 and dominant, therefore birational since char$(K) = 0$. We hope to piece together these rational parametrizations consistently, in order to parametrize an open subset of $\tilde{X}$.

The key to doing this is to find a rational section of the quadric bundle $\tilde{X} = \{Q_\Theta \mid \Theta \in \mathbb{P}^{n-l-1}\}$, i. e. a rational map $\sigma : \mathbb{P}^{n-l-1} \dashrightarrow \tilde{X}$ with $\sigma(\Theta) \in Q_\Theta$. If we could find such a section σ, then we would have a rational map

$$q_\Theta \mapsto \overline{\sigma(\Theta), q_\Theta} \in \mathbb{G}(1, \Theta), \tag{5}$$

defined for each Θ. These maps are now clearly compatible over $\tilde{X} - E$. Now, the spaces $\mathbb{G}(1, \Theta)$ form the "Grassmann bundle of lines" (which we will soon refer to as "Grass_1") in the projective bundle of $(l+1)$-planes Θ, parametrized by $\mathbb{P}^{n-l-1}$. Any Grassmann bundle over a rational base is rational (see Proposition 3.5, to follow). This rational map from $\tilde{X}$ into the Grassmann bundle of $\mathbb{G}(1, \Theta)$'s is 1-1 and dominant over each Θ, hence is birational overall. In short, this paragraph has shown:

Lemma 2.4. *The total space of a family of pointed quadrics over a rational base is rational.*

However, there is no guarantee of finding such a section σ, i. e. a choice of point $q_\Theta \in Q_\Theta$ varying regularly with $\Theta \in \mathbb{P}^{n-l-1}$. Say we take $l = 1$ and look only for points of Q_Θ that lie inside the line Γ : then in general $Z_\Theta := Q_\Theta \cap \Gamma$ will consist of two points, and we will have no way to pick one consistently. Thus we are forced to abandon hope of a *rational* parametrization, which is 1-1, and instead hope for a *unirational* parametrization, i. e. a dominant map $\mathbb{P}^N \dashrightarrow X$—or, equivalently, a dominant map from a rational variety to X. (Although in fact, smooth cubic surfaces in $\mathbb{P}^3$ are rational.)

Now, by analogy we hope to find a "*uni*rational section" of the bundle of the quadric bundle $\tilde{X} = \{Q_\Theta\}$, i. e. a map from a rational variety to $\tilde{X}$ that hits an open subset of $\Theta \in \mathbb{P}^{n-l-1}$. Fortunately, we will find such a rational variety by directly fixing the problem of the previous paragraph (using a familiar construction): consider the incidence correspondence

$$\Psi = \{(\Theta, p) \mid p \in Z_\Theta = Q_\Theta \cap \Gamma\} \subset \mathbb{P}^{n-l-1} \times \Gamma.$$

This variety Ψ manifestly has a "unirational section" to $\tilde{X}$, given by $(\Theta, p) \mapsto p \in Q_\Theta$. And Ψ is itself a rational variety, which we will prove shortly in Lemma 2.5.

What good is a "unirational section?" It furnishes a rational section not of $\tilde{X}$ but of a quadric bundle that dominates $\tilde{X}$, namely the pullback

$$H := \tilde{X} \times_{\mathbb{P}^{n-l-1}} \Psi.$$

This variety H is a quadric bundle over Ψ, whose fiber over (Θ, p) is by definition the pointed quadric $(p \in Q_\Theta)$. Thus by Lemma 2.4, H is rational if Ψ is rational. We can describe H more concretely as follows:

$$H = \{(q, \Theta, p) \mid q \in Q_\Theta, p \in Z_\Theta\} \subset \tilde{X} \times \mathbb{P}^{n-l-1} \times \Gamma.$$

So H clearly dominates $\tilde{X}$ and, pending the following result, we have obtained a unirational parametrization of $\tilde{X}$ and hence of X.

Lemma 2.5. *The incidence correspondence Ψ is rational.*

Proof. The variety $\Psi \subset \mathbb{P}^{n-l-1} \times \Gamma$ is (over an open subset) a bundle of $(l-1)$-dimensional quadric hypersurfaces $Z_\Theta = Y_\Theta \cap \Gamma \subset \Gamma$ over $\mathbb{P}^{n-l-1}$, so it seems at first that we have made no progress towards a proof. However, consider Ψ as as a bundle over its other factor, Γ : observe that a point $p \in \Gamma$ simply imposes a linear condition on $\Theta \in \mathbb{P}^{n-l-1}$, which corresponds to the tangent hyperplane $\mathbb{T}_p X$ to X at p. In fact, $\Psi = \{Z_\Theta\}_{\Theta \in \mathbb{P}^{n-l-1}}$ is in fact a *linear system* of hypersurfaces in $\Gamma \cong \mathbb{P}^l$, parametrized by $\Theta \in \mathbb{P}^{n-l-1}$. (This will be shown explicitly in Section 2.4.) Furthermore, this system has no basepoints, since a basepoint would be a singular point of X (see also 2.4). Therefore each fiber of the projection $\Psi \to \Gamma$ is a hyperplane in $\mathbb{P}^{n-l-1}$, so Ψ is a $\mathbb{P}^{n-l-2}$-bundle over $\Gamma \cong \mathbb{P}^l$ (by Proposition 3.3 to follow), hence rational. $\square$

2.3. **Constructions.** We proceed to introduce several objects of which we will make extended use in this chapter and the next. For the duration of Section 2.3, we let $E \to B$ be a vector bundle of rank $r + 1$, with B and E irreducible varieties (i. e. integral quasi-projective K-schemes, in our terminology).

2.3.1. *The relative Grassmannian.* It is possible to define the *relative Grassmannian variety* $\text{Grass}_k(E) \to B$, whose fiber over each closed point $b \in B$ is the ordinary Grassmannian $G(k+1, E_b)$—one can simply define $\text{Grass}_k(E)_U = U \times G(k+1, r+1)$ over each open subset $U \subset B$ such that $\pi^{-1}(U) \cong U \times K^{r+1}$, pending a compatibility check.

As mentioned already, we shall restrict our attention to the case that E is a sub-bundle of a trivial bundle $B \times V$, with V an $(n+1)$-dimensional vector space. Then we may simply define

$$\text{Grass}_k(E) = \{(b, \Lambda_b) \mid \Lambda_b \subset E_b\} \subset B \times G(k+1, V).$$

On each open subset $B \times U_\alpha$, with U_α an affine patch as in Proposition 1.1, $\text{Grass}_k(E)$ is defined as a subvariety by the requirement that each row of the matrix M_Λ lie within the linear space E_b.

Given a subbundle $E_0 \subset E$ of rank $r' \leq r$, we will also consider the subvariety $\text{Grass}_k(E; E_0) \subset \text{Grass}_k(E)$ of $(k+1)$-planes containing E_0, which is isomorphic to $\text{Grass}_{k-r'}(E/E_0)$. If $P = \mathbb{P}(E)$ is the projective bundle corresponding to the vector bundle E, then we will write $\text{Grass}_k(P)$ for the variety of projective k-planes over B, which is identical to $\text{Grass}_k(E)$. We will also let $\text{Grass}_k(P; P_0) = \text{Grass}_k(E; E_0)$ be the variety of projective k-planes contained in P and containing $P_0 = \mathbb{P}(E_0)$.

Definition 2.6. In general, we will refer to any B-scheme of the form $\text{Grass}_k(E) \to B$, for some vector bundle $E \to B$, as a *Grassmann bundle.*

2.3.2. *The relative Fano variety.* Given a closed subscheme $X \subset P = \mathbb{P}(E)$, it is also possible to define the *relative Fano variety* of $X \to B$,

$$F_k(X_{/B}) = \{(b, \Lambda_b) \in \text{Grass}_k(P) \mid \Lambda_b \subset X_b = X \cap E_b\}.$$

We show that $F_k(X_{/B})$ is a variety if E and B are varieties. Let b be a point of the variety B (i. e. a closed point). Then we can find an affine neighborhood U of b such that $\pi^{-1}(U) \cong U \times \mathbb{P}^r$ is trivial, and X_U is cut out by homogeneous polynomials on $\mathbb{P}^r$ whose coefficients are regular functions on U (we can choose finitely many such polynomials, and a neighborhood of b on which all of their coefficients are regular). Then for each of these homogeneous polynomials, the relative Fano variety $F_k(X_{/B})_U$ is merely a form of the incidence correspondence "I" of Section 1.2.1. The argument given there shows that $F_k(X_{/B})_U$ is a closed subvariety of $\text{Grass}_k(E)_U = U \times \mathbb{G}(k, r)$. (The case there was the universal family of hypersurfaces in a fixed projective space). Notice that we ignore the extra scheme-theoretic data from this definition: we consider the Fano *variety* as opposed to the Fano scheme.

Alternatively, if $P \subset B \times \mathbb{P}^n$ is a projective sub-bundle over B and $X \subset P$ a closed subvariety, then we may define the relative Fano variety simply as the subvariety of the absolute Fano variety $F_k(X)$ consisting of planes of the form $\{b\} \times \Lambda \subset B \times \mathbb{P}^n$, i. e. those contained in the first ruling of the Segre embedding (as per [13] Theorem 9.22). In fact, the original "I" could also have been defined as such.

2.3.3. *Some line bundles and Cartier divisors.* We will interpret some basic objects and describe how they carry over to the context of projective bundles.

The sheaves $\mathscr{O}_P(d)$. We will briefly describe how to extend the definition of the familiar sheaf $\mathscr{O}_{\mathbb{P}^n}(d)$ to a general projective bundle $P = \mathbb{P}(E)$. The sections of the resulting sheaf

$\mathscr{O}_P(d)$ $(d > 0)$ over an open subset $B_1 \subset B$ such that $E|_{B_1} \cong B_1 \times V$ are homogeneous forms of degree d on V with coefficients regular over B_1. Thus we may conclude that $\mathscr{O}_P(d)(P_{B_1}) = \mathrm{Sym}^d E^*(B_1)$, and therefore $\pi_*(\mathscr{O}_P(d)) = \mathrm{Sym}^d E^*$. A global section of $\mathscr{O}_P(d)$ defines a closed subscheme of $P \to B$. [6]

Let $S \subset \mathbb{P}V \times V$ be the universal line bundle over $\mathbb{P}V = G(1, V)$—for each $\ell \in \mathbb{P}V$, S has fiber $S_\ell = \ell \subset V$. Define $\mathscr{O}(-1) = \mathscr{O}_{\mathbb{P}^r}(-1)$ as the sheaf of sections of S, and define the "tautological sheaf" $\mathscr{O}(1)$ as the sheaf of sections of S^* (see [15] vol. 2, ch. VI for a complete description). We may also define $\mathscr{O}(d) = \mathscr{O}(1)^{\otimes d}$.

Similarly, define $S_P \to P$ to be the universal line bundle over $P = \mathrm{Grass}_0(E)$, i. e. the sub-bundle of $P \times_B E$ which has fibers $(S_P)_{(b,\ell_b)} = \ell_b \subset E_b$. We can now define $\mathscr{O}_P(-1)$ to be the sheaf of sections of S_P, $\mathscr{O}_P(1)$ to be the sheaf of sections of S_P^*, and $\mathscr{O}_P(d) = \mathscr{O}_P(1)^{\otimes d}$.

Now, let $B_0 \subset B$ be an affine open subset such that $E|_{B_0} \cong B_0 \times V$ and $P|_{B_0} \cong B_0 \times \mathbb{P}V$. We emulate the argument of [15] vol. 2, VI.1.4, Example 2, to find the sections of $\mathscr{O}_P(1)$ over $B_0 \times \mathbb{P}V$. One can see from the definitions that $S_P|_{P_{B_0}} = B_0 \times \mathbb{P}V \times_{\mathbb{P}V} S$, which implies $S_P^*|_{P_{B_0}} = B_0 \times \mathbb{P}V \times_{\mathbb{P}V} S^*$. Choose coordinates $\{X_\alpha\}$ on $\mathbb{P}V$, and consider the affine coordinate patches $U_\alpha = \{X_\alpha \neq 0\} \subset \mathbb{P}V$. Exactly as in [15], the line bundle $S_P^* \to \mathbb{P}V$ is trivial over each open set $B_0 \times U_\alpha$, with transition functions $c_{\alpha\beta} = X_\alpha / X_\beta$ between them. Sections over $B_0 \times U_\alpha$ are regular functions: these are of the form $\varphi_\alpha = P_\alpha / X_\alpha^d$ with P_α a homogeneous form of degree d in the X_α with regular functions on B_0 as coefficients. For such a section to be regular on $B_0 \times U_\beta$ for all β, we must have $c_{\alpha\beta}\varphi_\alpha = X_\alpha P_\alpha / (X_\beta X_\alpha^d)$ regular on U_β, which implies $d = 1$. Thus the sections of S_P^* over $P_{B_0} = B_0 \times \mathbb{P}V$, i. e. the elements of $\mathscr{O}_P(1)(B_0 \times \mathbb{P}V)$, are homogeneous linear forms with regular functions on B_0 as coefficients. This description then clearly extends to non-affine open subsets $B_1 \subset B$ over which P is trivial. Likewise, the sections of $\mathscr{O}_P(d)$ over B_1 are forms of degree d. These homogeneous linear forms take values on E, so we may conclude that $\pi_*(\mathscr{O}_P(d)) = \mathrm{Sym}^d E^*$ for $d > 0$.

Cartier divisors. We will think of a Cartier divisor D on a scheme X as a global section of the sheaf $\mathscr{M}_X^* / \mathscr{O}_X^*$ (defined in [17] II.6), which agrees exactly with the notion of a "locally principal divisor" on an irreducible variety in the classical setting. With this definition, a Cartier divisor D is effective iff it can be represented on an open cover $\{U_\alpha\}$ by elements of $\mathscr{O}_X(U_\alpha) \subset \mathscr{M}_X(U_\alpha)$, called "local equations" for D. An effective Cartier divisor D thus defines a locally principal ideal sheaf $\mathscr{I}_D$ and hence a closed subscheme of X, referred to by the same letter D. In [15] vol. 2, VI.1.4, classes of Cartier divisors are shown to be equivalent to invertible sheaves and line bundles, which allows us the pushforward and pullback operations, respectively (though one may lose local triviality in applying the pushforward). The Cartier divisors manifestly form a group.

In the case of the projective bundle $P = \mathbb{P}(E)$, one can see from the previous description that the nonzero global sections of $\mathscr{O}_P(d)$, for all $d \geq 0$, form a graded submonoid of the effective Cartier divisors. Furthermore, if $\alpha \in H^0(P, \mathscr{O}_P(d))$ and $\beta \in H^0(P, \mathscr{O}_P(d'))$ define subschemes $P_\alpha \subset P_\beta \subset P$, respectively, then β/α will be a global section of $\mathscr{O}_P(d' - d)$.

[6] Note that the vector bundles $\mathscr{O}_P(d)$, $d > 0$, are not necessarily sub-bundles of trivial bundles, or even projectively embeddable; but this will remain true of all *projective* bundles we will consider, thereby not violating Remark 2.2.

To see this quickly, consider the subschemes of E defined by α and β, meaning the cones $\hat{P}_\alpha \subset \hat{P}_\beta$ over $P_\alpha \subset P_\beta$ respectively. Over any affine open set $B_0 \subset B$ as above, the homogeneous polynomials representing α and β are regular functions on $E|_{B_0}$—so the scheme-theoretic inclusion $\hat{P}_\alpha \subset \hat{P}_\beta$ implies that $\alpha | \beta$ as homogeneous polynomials on each $B_0 \times \mathbb{P}V$. We will not, however, need the full facts of the case, i. e. that the group of line bundles $\mathscr{O}_P(d)$ represents a subgroup of the class group, etc.

2.3.4. *Residual varieties.* The goal of this section is to formalize and generalize the "residual" intersections involved in the proof of the unirationality of cubic hypersurfaces, Proposition 2.3.

Definition 2.7. A *family of hypersurfaces* over B is a closed subscheme $X \subset P = \mathbb{P}(E)$ defined by the vanishing of a global section s_X of $\mathscr{O}_P(d)$ that does not vanish identically over any point of B. By a family of *l-planed* hypersurfaces, we mean a triple of B-schemes (Γ, X, P) such that $\Gamma = \mathbb{P}(E_0)$ for a sub-bundle $E_0 \subset E$, and $\Gamma \subset X \subset P$.

By the previous discussion, 2.3.3, a family of hypersurfaces is an effective Cartier divisor on P. We define several projective bundles associated to the bundle P. Write

$$\Pi = \mathrm{Grass}_{l+1}(P; \Gamma)$$

for the family of projective $(l+1)$-planes containing Γ (such a plane was referred to as Θ in the proof of Proposition 2.3). Π is then a $\mathbb{P}^{n-l-1}$-bundle over B; it is in fact locally trivial, by Lemma 3.3 to follow. Let $\tilde{\Pi} \subset P \times_B \Pi$ be the universal bundle over Π, whose fiber over a closed point of Π is the corresponding projective $(l+1)$-plane in P. Consider the pullback bundles $\Gamma_\Pi = \Gamma \times_B \Pi$, and $P_\Pi = P \times_B \Pi$. Then we have the inclusions

$$\Gamma_\Pi \subset \tilde{\Pi} \subset P_\Pi.$$

Note that these are projective bundles over the base Π, itself a $\mathbb{P}^{n-l-1}$-bundle over the original base B.

Now, let $X \subset P$ be a family of hypersurfaces over B, defined by a global section s_X of $\mathscr{O}_P(d)$. The pullback X_Π is again an effective Cartier divisor, in P_Π, defined again by s_X (viewed as a section of $\mathscr{O}_\Pi(d)$). Consider the scheme-theoretic intersection

$$X_\Pi \cap \tilde{\Pi} \subset \tilde{\Pi}.$$

This is yet again a Cartier divisor, defined by the restriction to $\tilde{\Pi}$ of s_X which is in turn a section of $\mathscr{O}_{\tilde{\Pi}}(d)$. This restriction does not vanish identically: for, $\tilde{\Pi}$ is irreducible (being a vector bundle over a projective bundle over the irreducible variety B), and X_Π does not contain any fiber $\tilde{\Pi}_b$ over $b \in B$ (the latter projects surjectively onto $P_b \cong \mathbb{P}^r$, while the former does not).

Finally, note that the family of l-planes $\Gamma_\Pi \subset \tilde{\Pi}$ is an effective Cartier divisor, for which a defining section of $\mathscr{O}_{\tilde{\Pi}}(1)$ can readily be described locally on $\tilde{\Pi}$.

Definition 2.8. Corresponding to our given family of l-planed hypersurfaces (Γ, X, P) over B, we have several constructions:

(a) The *main residual Π-scheme* will refer to the effective Cartier divisor

$$Y := X_\Pi \cap \tilde{\Pi} - \Gamma_\Pi \subset \tilde{\Pi}.$$

By the preceding discussion, Y is defined by a section of $\mathscr{O}_{\tilde{\Pi}}(d-1)$.
(a') Define the *locus of definition* of Y, $\mathrm{DEF}(Y) \subset \Pi$, to be the maximal open subset of $(l+1)$-planes $\Theta \in \Pi$ for which $Y_\Theta \subset \tilde{\Pi}_\Theta$ has dimension l, i. e. is a hypersurface. If $\mathrm{DEF}(Y)$ is non-empty, then

$$Y_{\mathrm{DEF}(Y)} \to \Pi,$$

will be called the *residual family of hypersurfaces* attached to (Γ, X, P).

(b) The *secondary residual* Π*-scheme* Z will refer to the scheme-theoretic intersection

$$Z := Y \cap \Gamma_\Pi.$$

(b') Write $\mathrm{DEF}(Z) \subset \Pi$ for the maximal open subset over which $Z_{\mathrm{DEF}(Z)} \to \Pi$ has fiber dimension $l-1$, i. e. is a hypersurface.

To give a very rough summary: the first of these, Y, is the family of intersections of X with the family Π of $(l+1)$-planes containing the family of l-planes $\Gamma \subset X$, residual to Γ. The second, $\mathrm{DEF}(Y) \subset \Pi$, is the open subset having nontrivial fibers in Y. The third, Z, is the family of intersections of Y with the base family of l-planes. Notice that $\mathrm{DEF}(Z) \subset \mathrm{DEF}(Y)$.

2.4. **Explicit description of the fibers of the residual varieties.** Over an individual closed point $b \in B$, the fibers of the residual varieties are readily describable as subvarieties of $P_b \cong \mathbb{P}^r$. To do this, we work temporarily over the base $B = \mathrm{Spec}(K)$. Choose homogeneous coordinates $\{V_0, \dots, V_l, W_{l+1}, \dots, W_r\}$ on $P = \mathbb{P}^r$ so that $\Gamma = \mathbb{P}^l = \{W_i = 0\}$. Then $\{W_i\}$ are homogeneous coordinates on $\Pi = \mathbb{P}^{n-l-1}$. Write the defining equation of $X (= X_b)$ as

$$F(V, W) = \sum_{0 \le |I| \le d-1} V^I F_I(W).$$

Here each I is a multi-index of degree $|I|$, and each $F_I(W)$ is a homogeneous polynomial of degree $d - |I| \ge 1$. Now, given a point $[w_{l+1}, \dots, w_r] \in \Pi$, we may parametrize the corresponding plane as

$$\Theta = \{[V_0, \dots, V_l, w_{l+1}U, \dots, w_r U]\}.$$

We thus have homogeneous coordinates $[V_0, \dots, V_l, U]$ on Θ such that $\Gamma \subset \Theta$ is the plane defined by $U = 0$. The restriction of F to the $(l+1)$-plane Θ is given by

$$F(V, W)|_\Theta = \sum V^I U^{d-|I|} F_I(w_{l+1}, \dots, w_r).$$

Since $0 \le |I| < d$, we may divide through by U to obtain the defining equation for the primary residual Π-scheme Y,

$$F_Y = F(V, W)|_\Theta / U = \sum F_I(w_{l+1}, \dots, w_r) U^{d-|I|-1} Z^I.$$

The scheme-theoretic intersection of Y with Γ is given simply by setting $U = 0$, by which we obtain the defining equation for the secondary residual scheme Z,

$$F_Z(V, \Theta) = \sum_{|I'| = d-1} F_{I'}(w_{l+1}, \dots, w_r) Z^{I'}.$$

Here, the $F_{I'}$ are homogeneous linear forms. From these equations, we see that the locus $\mathbb{P}^{n-l-1} - \mathrm{DEF}(Y)$, i. e. the set of Θ which are equal to Y_Θ, is the common zero locus of the homogeneous forms F_I. Likewise, the locus $\mathbb{P}^{n-l-1} - \mathrm{DEF}(Z)$ is the common zero locus of the linear forms $F_{I'}$ and thus a subset of the former; hence $\mathrm{DEF}(Z) \subset \mathrm{DEF}(Y)$.

From this discussion, we conclude the following:

Proposition 2.9. *Let (Γ, X, P) be an l-planed family of hypersurfaces over an integral base B, and let $b \in B$ be a closed point. Then,*

(a) *If Γ_b meets the smooth locus of the morphism $\pi : X \to B$ (i. e. the smooth locus of X_b), then $DEF(Z) \cap \Pi_b \neq \emptyset$.*
(b) *Let $\mathbb{P}^p \subset \Pi_b$ be the common zero locus of the linear forms $F_{I'}$ coming from the terms of F_Z. Then the secondary residual schemes*

$$\{Z_{\Theta_b}\}$$

form a linear system of hypersurfaces in $\Gamma_b \cong \mathbb{P}^l$ parametrized by the quotient of Π_b by the subspace $\mathbb{P}^p$, i. e. $\mathbb{P}^{r-p-1}$. The basepoint locus of this system is contained in the singular locus of the morphism $\pi : X \to B$.

Remark 2.10. For our purposes, the "smooth locus" of the map π is the dense open subset of X on which $\dim(\ker(d\pi))$ attains a Zariski-local minimum (see [13], ch. 14). So, by Proposition 3.2 (to follow), a closed point $x \in X$ lies in the smooth locus of π if and only if the scheme-theoretic fiber $X_{\pi(x)}$ through x is smooth at x. Thus the requirement of Proposition 2.9 is simply that for each point $b \in B$, Γ_b not be contained in the singular locus of the hypersurface $X_b \subset P_b$.

Proof. Assume that the constructions of this section have been made for the given closed point $b \in B$, i. e. $F(V, W)$ defines $X_b \subset P_b$, the V_i are coordinates on Γ_b, and so-on.

(a) By the remark, we know that $\Gamma_b \not\subset (X_b)_{\mathrm{sing}}$. The V-partials of $F(V, W)$ are all zero on $\Gamma_b = \{W_i = 0\}$. A W-partial along Γ_b is

$$\left.\frac{\partial F(V, W)}{\partial W_i}\right|_{\Gamma_b} = \sum_{|I'|=d-1} V^{I'} F_{I'}(0, \ldots, 1, \ldots, 0) = F_Z(V, 0, \ldots, 1, \ldots, 0),$$

with a 1 in the i'th argument of each. So, $\frac{\partial F}{\partial W_j}(V, 0) \neq 0 \Rightarrow F_Z(V, 0, \ldots, 1, \ldots, 0) \neq 0 \Rightarrow [0, \ldots, 1, \ldots, 0] \in \mathrm{DEF}(Z)_b$. Hence $(X_b)_{sm} \neq \emptyset \Rightarrow \mathrm{DEF}(Z)_b \neq \emptyset$

(b) By the definition of F_Z, the hypersurfaces Z_{Θ_b} manifestly form a linear system of degree $d-1$ in Γ_b parametrized by the quotient $\Pi_b/\mathbb{P}^p$. A basepoint $(V', 0)$ of this system would yield a solution, for all W, to the equation

$$0 = F_Z(V', W) = \sum_{|I'|=d-1} V^{I'} F_{I'}(W) = \sum W_i \frac{\partial F}{\partial W_i}(V', 0).$$

This would imply $\frac{\partial F}{\partial W_i}(V', 0) = 0$ for all i, meaning that V' is a singular point of X_b. Thus the basepoint locus of $\{Z_{\theta_b}\}$ is contained in the singular locus of X_b. $\square$

2.5. **Numbers.** Finally, we introduce the relevant numerical bounds of which we will make use in the following section. Define

$$M(d,l) = \binom{l+d-1}{d-1} + l - 1.$$

This number has the following use:

Lemma 2.11. *Let d, $l \geq k \geq 0$, and $n \geq M(d,l)$ be non-negative integers, and let $\Lambda \subset X \subset \mathbb{P}^n$ be a k-plane contained in a hypersurface X of degree d. Then there exists an l-plane Γ such that $\Lambda \subset \Gamma \subset X$.*

Proof. Since $M(d,l') \leq M(d,l)$ for all $l' \leq l$, it is sufficient to assume $k = l-1$. Choose homogeneous coordinates $V_0, \dots, V_k, W_{k+1}, \dots, W_n$ on $\mathbb{P}^n$ such that the k-plane $\Lambda = \{W_i = 0\}$. In these coordinates, the defining polynomial of X is of the form

$$F(V,W) = \sum_{0 \leq |I| < d} V^I F_I(W_{k+1}, \dots, W_n).$$

This sum ranges over the $\binom{k+1+d-1}{d-1} = \binom{k+d}{k+1}$ monomials V^I of degree less than $d-1$, and each F_I are homogeneous of degree $d - |I| \geq 1$ in the W_i.

Now, given that $n - k \geq M - k \geq \binom{k+d}{k+1}$, the polynomials F_I have a nontrivial common root $[W_{k+1}, \dots, W_n]$. Thus the span, Γ, of Λ and the point $[0, \dots, 0, W_{k+1}, \dots, W_n]$ is an l-plane such that $\Lambda \subset \Gamma \subset X$, as desired. □

Next we fix an integer $k \geq 0$ and define two functions recursively, $N_0(d,k)$ and $N(d,k)$. Begin by setting

$$N(2,k) = N_0(2,k) = \binom{k+1}{2} + 3.$$

Then for $d \geq 3$, define recursively

$$N_0(d,k) = M(d, N(d-1,k)+1)$$

and then

$$N(d,k) = N_0(d,k) + \binom{k+d}{d} + 2,$$

or in other words,

$$N_0(d,k) = M(d, N_0(d-1,k) + \binom{k+d-1}{d-1} + 3).$$

Note that both of these functions are strictly increasing with k.

Proposition 2.12. **(a)** *In the case $d = 2$, note that $N_0(2,k) \geq M(2,k)$. So for $n \geq N_0(2,k)$. So a quadric hypersurface $Q \subset \mathbb{P}^n$ is swept out by k-planes. This in turn implies that if Q is smooth, then*

$$\dim(F_k(Q)) = \phi(n,2,k).$$

(b) *For all $d \geq 3$ and $k \geq 0$,*

$$N_0(d,k) \geq \binom{k+d}{d} + 3k + 1 \geq M(d,k),$$

and therefore a hypersurface $X \subset \mathbb{P}^n$ with $n \geq N_0(d,k)$ is swept out by k-planes.

Proof. *(a)* For $d = 2$, note that $M(2,k) = k + 1 + k - 1 = 2k \leq \frac{k+1}{2}k + 3$, which is the first claim. So by Lemma 2.11, any quadric in $\mathbb{P}^n$, $n \geq N_0(d,k)$, is swept out by k-planes. In particular, this implies that the map $\pi_1 : I \to \mathbb{P}^N$ from Chapter 1 is surjective: so for a general quadric hypersurface Q, the arguments of Chapter 1 imply that $\dim(F_k(Q)) = \dim(I) - \dim(\mathbb{P}^N) = \phi(n,2,k)$. However, a general quadric is smooth, and all smooth quadrics are projectively equivalent, so this applies to all smooth quadrics.

(b) Fix k. We first do the base case $d = 3$:

$$N_0(3,k) = \binom{\binom{k+1}{2} + 3 + \binom{k+2}{2} + 2 + 3}{2} \geq \binom{(k+1)^2 + 8}{2} \geq \binom{k+3}{3} + 3k + 1$$

by comparing coefficients.

Now, for a given $d \geq 3$ and $k > 1$, assume the result for all (d', k') such that $d' < d$ or $k' < k$. Then,

$$\begin{aligned} N_0(d,k) \geq M(d, 2\binom{k+d-1}{d-1} + 3k + 4) \quad &\geq \quad \frac{2(k+d-1)}{d-1}(2\binom{k+d-1}{d-1} + 3k + 4 - 1) \\ &\geq \quad 2\binom{k+d}{d} + 3k + 3. \quad \square \end{aligned}$$

2.6. **Low-degree smooth hypersurfaces do not have too many k-planes.**

Theorem 2.13. *Let n, d, k be positive integers, and let $N_0 = N_0(d,k)$ as defined above. If $n \geq N_0(d,k)$ and $X \subset \mathbb{P}^n$ is a smooth hypersurface, then*

$$\dim(F_k(X)) = \phi(n,d,k).$$

We will need two corollaries during the induction step of the proof. (In the next chapter, we will in fact need two further corollaries to this theorem; but these will be deferred until section 3.4.2.)

Corollary 2.14. **(a)** *If the codimension of X_{sing} in the hypersurface X is at least N_0, then we may likewise conclude*

$$\dim(F_k(X)) = \phi(n,d,k).$$

(b) *Let $N(d,k)$ as defined earlier, and let $X \subset \mathbb{P}^n$ be any hypersurface of degree d. If $n \geq N(d,k)$, then*

$$\dim(F_k(X)) \leq \max\{\phi(n,d,k), \phi(n,d,k) + \dim(X_{sing}) - 1\}.$$

Proof of Corollary 2.14. *(a)* Assume the theorem for a given d and k. For $n < N_0$ the desired corollary is vacuous, and for $n = N_0$ it is just the theorem. We will prove the cases $n > N_0$ by induction on n—so let $n > N_0$ and $X \subset \mathbb{P}^n$ be a hypersurface of degree d whose singular locus has codimension at least N_0. Let $H \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n$ be a general hyperplane, and consider $\mathbb{G}(k,H) \cong \mathbb{G}(k,n-1) \subset \mathbb{G}(k,n)$, the sub-Grassmannian of k-planes contained in H. Set $Y = X \cap H$. Then by Bertini's Theorem, the singular locus of Y again has codimension at least N_0, so by induction $\operatorname{codim}(F_k(Y) \subset \mathbb{G}(k,n)) = \binom{k+d}{d}$. But $F_k(Y) = F_k(X) \cap \mathbb{G}(k,H) \subset \mathbb{G}(k,n)$; so, by the subadditivity of codimension of intersections in the smooth variety $\mathbb{G}(k,n)$ ([13] 17.24), if a component F of $F_k(X)$ meets $\mathbb{G}(k,H)$ then

(6)

$$\operatorname{codim}(F \subset \mathbb{G}(k,n)) \geq \operatorname{codim}(F_k(X) \subset \mathbb{G}(k,n)) \geq \operatorname{codim}(F_k(Y) \subset \mathbb{G}(k,H)) = \binom{k+d}{d}.$$

It remains to show that every component of $F_k(X)$ meets $\mathbb{G}(k,H)$. This will follow from a standard result in the intersection theory of Grassmannians, which we state without proof:

Lemma 2.15. *If X_1 and X_2 are subvarieties of $\mathbb{G}(k,n)$ of codimension c_1 and c_2 such that $c_1 + c_2 < n+1-2k$, then the intersection $X_1 \cap X_2$ is nonempty.*

Note that in the case $k=0$, this gives the standard intersection criterion on $\mathbb{P}^n$.

Now, recall that $F_k(X)$ is cut out by $\binom{k+d}{d}$ conditions, hence any component F_0 has codimension at most that number. So the codimensions of F_0 and $\mathbb{G}(k,H) \subset \mathbb{G}(k,n)$ are at most $\binom{k+d}{d}$ and $k+1$, respectively. We have

$$n+1-2k \geq N_0 + 1 - 2k \geq \binom{k+d}{d} + 3k + 2 - 2k = \binom{k+d}{d} + k + 2$$

by Proposition 2.12(b), and so by the lemma F meets $\mathbb{G}(k,H)$, which establishes 6 and the claim.

(b) Assume the result of part (a) for the given pair d,k, and let X be a hypersurface of degree d in $\mathbb{P}^n$, with $n \geq N(d,k)$.

If $\operatorname{codim}(X_{\text{sing}} \subset X) \geq N_0$ then part (a) applies; otherwise $\dim(X_{\text{sing}}) \geq n - N_0 \geq N(d,k) - N_0 = \binom{k+d}{d} + 2$, by definition of $N(d,k)$. So,

$$\begin{aligned} \dim(F_k(X)) \quad &\leq \mathbb{G}(k,n) \\ &= \phi(n,d,k) \quad + \binom{k+d}{d} \\ &\leq \phi(n,d,k) \quad + \dim(X_{\text{sing}}) - 1. \end{aligned}$$

□

Remark 2.16. For any given n,d,k, these corollaries can be deduced from the theorem as applied only to these n,d,k. Thus we will be able to use the corollaries in conjunction with our induction hypothesis.

2.6.1. *A Bertini Lemma.* We will need the following lemma to control the singularities of our hypersurfaces.

Lemma 2.17. (Bertini's Theorem for a projective space) *Let $\mathscr{D} = \{D_\alpha\}_{\alpha \in \mathbb{P}^m}$ be a linear system of hypersurfaces in $\mathbb{P}^n$, with base locus $B \subset \mathbb{P}^n$ of dimension b. Define the subsets*

$$S_k = \{D_\alpha \in \mathscr{D} \mid \dim((D_\alpha)_{sing}) \geq b+k\}.$$

Then S_k is a projective variety and

$$codim(S_k \subset \mathbb{P}^m) \geq k.$$

Proof. Define the incidence correspondence

$$\Sigma = \{(\alpha, p) \in \mathbb{P}^m \times \mathbb{P}^n \mid p \in (D_\alpha)_{\text{sing}}\}.$$

Then $S_k = \{\alpha \in \mathbb{P}^m \mid \dim(\pi_1^{-1}(\alpha)) \geq k + b\}$, which is a projective variety in $\mathbb{P}^m$ since fiber dimension is an upper-semicontinuous function of $\alpha \in \mathbb{P}^m$ (Proposition 1.3(b)).

We may restrict π_1 to be surjective onto S_k, to obtain the relation

$$\begin{aligned} \dim(S_k) + k + b &= \dim(\pi_1^{-1}(S_k)) \\ &\leq \dim(\Sigma), \end{aligned}$$

so we will be done upon establishing

$$\dim(\Sigma) \leq m + b. \tag{7}$$

We will examine the projection π_2 to obtain the bound in (7) on $\dim(\Sigma)$. Consider the map $f : \mathbb{P}^n - B \to (\mathbb{P}^m)^*$ associated to the linear system $\mathcal{D}$. Explicitly, let the vector space of meromorphic functions underlying $\mathbb{P}^m$ have basis $\{f_0, \ldots, f_m\}$, so we may write $f(p) = [f_0(p), \ldots, f_m(p)]$. Since p is outside the base locus B, we may assume $f_0(p) \neq 0$ so we can take affine coordinates $z_1, \ldots, z_m$ with which to write

$$(df_p)_{ij} = \left.\frac{\partial(f_i/f_0)}{\partial z_j}\right|_p.$$

Conveniently, the rank l of this matrix is by definition the number of linear conditions for a hypersurface $D_\alpha \ni p$ corresponding to $\alpha = [a_0 f_0 + \ldots + a_m f_m]$ to be singular at the point p. Thus if $rk(df_p) = l$ then the space of hypersurfaces singular at p, which is the inverse image of p under π_2, is a linear subspace $\pi_2^{-1}(p) \cong \mathbb{P}^{m-l-1} \subset \mathbb{P}^m$.

Define the locally closed subset $W_l = \{p \in \mathbb{P}^n - B \mid rk(df_p) = l\}$. We have just determined that the fiber of π_2 over a point of W_l has dimension $m - l - 1$, so

$$\dim(W_l) + (m - l - 1) \geq \dim(\pi^{-1}(W_l)).$$

Since Σ is a union of the finitely many inverse images $\pi_2^{-1}(W_l)$ along with $\pi_2^{-1}(B)$, we must show that

$$\begin{aligned} m + b &\geq \dim(W_l) + m - l - 1 && (8) \\ b + l + 1 &\geq \dim(W_l). && (9) \end{aligned}$$

We will in fact be done once we show this inequality, because (trivially) the inverse image $\pi_2^{-1}(B)$ has dimension at most $m + b$.

We have translated a question concerning the singular loci $(D_\alpha)_{\text{sing}}$ into a question of the rank of df_p in order to apply Sard's theorem. This result ([13] Proposition 14.4) states that over a field of characteristic 0, df_p is surjective for points landing in a nonempty open subset of $f(W_l)$; so clearly

$$\dim(f(W_l)) \leq l.$$

This now implies

$$\dim(W_l) \leq \dim(f^{-1}(f(p))) + l,$$

for any fiber through a point $p \in W_l$. But the fibers of f all have dimension less than $b+1$. In fact, any non-constant map on the complement of a b-dimensional variety of projective space has fiber dimension at most $b + 1$, by the following argument:

Write $f = [F_0, \dots, F_m]$ so that the F_i are homogeneous polynomials with no common factor which are simultaneously zero nowhere outside of B (See [15] vol. I ch. 3). The projective variety defined by the F_i has dimension at most b. The fiber over (say) $p = [1, 0, \dots, 0]$ then is defined by $F_1 = \cdots = F_m = 0$. The section of this fiber by the hypersurface $\{F_0 = 0\}$ has dimension one less (since the F_i have no common factor), and is contained in B. So the fiber over p, defined by $F_1 = \cdots = F_m = 0$, has dimension at most $b + 1$.

This establishes (9), which implies (7) and hence the main inequality of the lemma. □

2.6.2. *Proof of Theorem 2.13.* Theorem 1.6(c) of Chapter 1 asserts that in case $\phi(n, d, k) > 0$ and $d \geq 3$, every hypersurface contains a family of k-planes of dimension *at least* ϕ (the proof invoked the result of [1]). So it remains to show that a smooth hypersurface of sufficiently high dimension has a family of k-planes of dimension *at most* ϕ.

To this end, we will construct a convenient family of pairs of planes, Δ. We will then find a lower bound L (step 1) and an upper bound U (step 2) on $\dim(\Delta)$. The resulting inequality $L \leq U$ will simplify to the desired upper bound on $\dim(F_k(X))$ (step 3).

The bounds L and U will be established by induction on d. Proposition 2.12(a) exactly establishes the base case $d = 2$ of the theorem, for all k. For the induction (the remainder of the proof), fix d and k, and assume that Theorem 2.13 and its corollaries hold for all triples (n', d', k') with $n' \leq n, d' \leq d, k' \leq k$ but not all equal. Note that the theorem is vacuous for $n' < N_0(d', k')$. Let $X \subset \mathbb{P}^n$, with $n \geq N_0(d, k)$, be a smooth hypersurface of degree d.

Set $l = N(d-1, k)$, and define the locally closed subvariety $\Delta \subset F_k(X) \times F_l(X)$ given by

$$\Delta = \{(\Lambda, \Gamma) \in F_k(X) \times F_l(X) \mid \dim(\Lambda \cap \Gamma) = k - 1\}.$$

Equivalently, Δ can be defined by the condition that Λ and Γ together span an $(l+1)$-plane in $\mathbb{P}^n$ (not necessarily contained in X).

Remark 2.18. The idea is roughly as follows: our numbers are sufficiently high that not only will X be swept out by k-planes, but each of these k-planes in X is contained in an $l = N(d-1, k)$-plane in X. This will allow us to apply our induction hypothesis on d first to show that Δ is dense in a fiber product of flag-Fano varieties (step 1), then to inject Δ into the relative Fano variety of the main residual scheme Y coming from the intersection of X with the variety of $(N(d-1, k) + 1)$-planes containing its Fano variety of $l = N(d-1, k)$-planes (step 2).

Step 1. Let $F_{k-1,k}(X)$ and $F_{k-1,l}(X)$ be the flag-Fano varieties

$$F_{k-1,k}(X) = \{(\Omega, \Lambda) \mid \Omega \subset \Lambda \subset X\} \subset \mathbb{G}(k-1, n) \times \mathbb{G}(k, n)$$

and

$$F_{k-1,k}(X) = \{(\Omega, \Gamma) \mid \Omega \subset \Gamma \subset X\} \subset \mathbb{G}(k-1, n) \times \mathbb{G}(l, n).$$

We will realize Δ as an open, dense subset of the fiber product

$$\Phi := F_{k-1,k}(X) \times_{F_{k-1}(X)} F_{k-1,l}(X)$$

From the definition, there is a regular map

$$\Delta \to F_{k-1}(X)$$

given by

$$(\Lambda, \Gamma) \to \Omega := \Lambda \cap \Gamma.$$

Via this map, embed $\Delta \subset \Phi$ via the identification $(\Lambda, \Gamma) \mapsto ((\Omega, \Lambda), (\Omega, \Gamma)) \in \Phi$. This embedding is open, since $\Delta \subset \Phi$ is defined by the open condition $\Lambda \not\subset \Gamma$.

We claim that Δ is dense in the fiber product Φ. A point of $\Phi - \Delta$ can be represented by a triple $(\Omega \subset \Lambda \subset \Gamma)$. Since $n \geq N_0(d,k) = M(d, N(d-1,k)+1) = M(d, l+1)$, we can choose an $(l+1)$-plane $\Sigma \subset X$ such that $\Gamma \subset \Sigma$. Then the k-plane $\Lambda \subset \Gamma$ can be realized as a limit of k-planes $\Lambda_x \not\subset \Gamma$—simply choose a line ℓ that meets Γ in a single point $x_0 \in \Lambda - \Omega$, and set $\Lambda_x = \overline{\Lambda, x}$, for points $x \in \ell$. Thus $\dim(\Delta) = \dim(\Phi)$.

We get a standard lower bound on $\dim(\Phi)$. Note that the fibers of the second projection $F_{k-1,k} \to \mathbb{G}(k,n)$ are isomorphic to $\mathbb{P}^{k*}$, so

$$\dim(F_{k-1,k}(X)) = k + \dim(F_k(X)).$$

Similarly, the fibers of $F_{k-1,l} \to \mathbb{G}(l,n)$ are equal to $\mathbb{G}(k-1,l)$, so we have

$$\dim(F_{k-1,l}(X)) = k(l-k+1) + \dim(F_l(X)).$$

We have assumed that $n \geq N_0(d,k) \geq M(d,l) \geq M(d,k)$, so we know that both $F_{k-1,k}(X)$ and $F_{k-1,l}(X)$ map surjectively onto $F_{k-1}(X)$. So by [13] 11.15,

$$\begin{aligned} \dim(\Delta) = \dim(\Phi) \geq & \quad \dim(F_{k-1,k}(X)) + \dim(F_{k-1,l}(X)) - \dim(F_{k-1}(X)). \\ = & \quad k + \dim(F_k(X)) + k(l-k+1) + \dim(F_l(X)) - \dim(F_{k-1}(X)). \end{aligned}$$

(*Note:* From the proof of this last fact, we have equality if Δ is irreducible.)

Since $N_0(d,k) \geq N_0(d,k-1)$, we have from the induction hypothesis

$$\dim(F_{k-1}(X)) = \phi(n,d,k-1) = k(n-k+1) - \binom{k+d-1}{k-1},$$

and the last inequality reduces to

$$\dim(\Delta) = \dim(F_{k-1}(X)) \geq k(l-n+1) + \binom{k+d-1}{k-1} + \dim(F_k(X)) + \dim(F_l(X)) =: L$$

Step 2. The idea here is to get an upper bound U on $\dim(\Delta)$ in terms of just $\dim(F_l(X))$, so that the expression $L \leq U$ will involve only $\dim(F_k(X))$ as an unknown quantity.

We will regard the ambient space $P = \mathbb{P}^n$ as a projective bundle over Spec$K$. Let $F = F_l(X)$, and let $\tilde{F} \to F$ be the universal projective bundle, whose fiber over a point $\Gamma \in F$ is the corresponding projective l-plane.

Consider the (rather trivial) family of l-planed hypersurfaces $(\tilde{F}, X_F, P_F)$. Form the $\mathbb{P}^{n-l-1}$-bundle $\Pi \to F$ corresponding to this family—recall from section 2.8 that the fiber of Π over a point $\Gamma \in F$ is the set of $(l+1)$-planes Θ containing Γ. Also form the main and secondary residual Π-schemes, respectively $Y \supset Z$. Recall that the relative Fano variety $F_k(Y_{/\Pi}) \subset \text{Grass}_k(\tilde{\Pi})$ is the variety of pairs (Λ, Θ_Γ) such that $\Lambda \subset Y_{\Theta_\Gamma})$, where Λ is a k-plane and $\Theta_\Gamma \supset \Gamma$ an $(l+1)$-plane.

Define the map of varieties

$$\begin{aligned} \Delta \quad &\to \quad F_k(Y_{/\Pi}) \\ (\Lambda, \Gamma) \quad &\mapsto \quad (\Lambda, \Theta_\Gamma), \end{aligned}$$

where $\Theta_\Gamma = \overline{\Lambda, \Gamma}$ is the $(l+1)$-plane they span. Note that $\Lambda \subset Y_{\Theta_\Gamma}$ as required: the open dense set $\Lambda - \Gamma \subset \Lambda$ is certainly contained in the residual variety Y_{Θ_Γ}, which is closed and hence contains Λ.

This map is clearly injective. So in what follows, we will simply compute an upper bound on the dimension of $F_k(Y_{/\Pi})$. This will furnish our upper bound U on the dimension of Δ.

To do this, fix an l-plane $\Gamma_0 \subset X$, and we will compute the dimension of the fiber of $F_k(Y_{/\Pi})$ over Γ_0. For this purpose, we need only consider the residual family over the single l-plane Γ_0, i. e.

$$Y_0 := \{Y_\Theta \mid \Theta \in \Pi_{\Gamma_0} = \mathbb{P}^{n-l-1}\}.$$

So we would like to compute the dimension of

$$F_k(Y_{/\Pi})_{\Gamma_0} = F_k(Y_{/\Pi_{\Gamma_0}}) = F_k(Y_{0/\mathbb{P}^{n-l-1}}).$$

The Y_Θ have degree $d-1$, so we hope to apply the induction hypothesis once we know something about their singularities.

Happily, since X is smooth, by Proposition 2.9 the hypersurfaces $\{Z_\Theta = Y_\Theta \cap \Gamma_0\}$ form a *base-point-free* linear series in Γ_0—and so by Proposition 2.17, the locus

$$S_\rho = \{\Theta \in \mathrm{DEF}(Z)_{\Gamma_0} \subset \mathbb{P}^{n-l-1} \mid \dim((Z_\Theta)_{\mathrm{sing}} \geq \rho - 1\}$$

has codimension at least ρ in $\mathbb{P}^{n-l-1}$. Hence, the variety

$$W_\rho = \{\Theta \in \mathrm{DEF}(Z)_{\Gamma_0} \subset \mathrm{DEF}(Y)_{\Gamma_0} \mid \dim((Y_\Theta)_{\mathrm{sing}} \geq \rho\} \subset S_\rho$$

must also have codimension at least ρ in $\mathbb{P}^{n-l-1}$.

Thus we can choose $\Theta \in \mathrm{DEF}(Z)_{\Gamma_0} - W_1$, and the hypersurface Y_Θ has at most isolated singularities. So, since $l = N(d-1,k)$, the induction hypothesis yields from Corollary 2.14 that

$$\dim(F_k(Y_\Theta)) = \phi(l+1, d-1, k).$$

We may thus conclude that the inverse image of $\mathrm{DEF}(Z)$ in the fiber $F_k(Y_{0/\mathbb{P}^{n-l-1}}) = F_k(Y_{/\Pi})_{\Gamma_0}$ has dimension $\phi(l+1, d-1, k) + n - l - 1$.

It remains to check that the inverse image of the p-dimensional projective linear space $(\mathrm{DEF}(Z)_{\Gamma_0})^c = \mathbb{P}^{n-l-1} - \mathrm{DEF}(Z)_{\Gamma_0}$ (as per Proposition 2.9) does not introduce components of larger dimension in $F_k(Y_{/\Pi})_{\Gamma_0}$. Since the series $\{Z_\Theta\}$ is base-point free, we must have $l+1 \leq \mathrm{codim}((\mathrm{DEF}(Z)_{\Gamma_0})^c \subset \mathbb{P}^{n-l-1})$, since this is the dimension of the linear system $\{Z_\Theta\}$. Meanwhile we have the trivial upper bound $\dim(F_k(Y_\Theta)) \leq \dim(\mathbb{G}(k, l+1))$, so the fiber dimension can jump from the generic dimension by at most $\dim(\mathbb{G}(k, l+1)) - \phi(l+1, d-1, k) = \binom{k+d-1}{d-1}$. But since the locus $(\mathrm{DEF}(Z)_{\Gamma_0})^c$ having possibly higher-dimensional fibers has codimension at least $l+1 \geq \binom{k+d-1}{d-1}$, these fibers cannot contribute a higher-dimensional component of $F_k(Y_{/\Pi})_{\Gamma_0}$. We can conclude that

$$\dim(F_k(Y_{/\Pi})_{\Gamma_0}) = \phi(l+1, d-1, k) + n - l - 1.$$

Since $\Gamma_0 \in F_l(X)$ was arbitrary, we may conclude finally that

$$\begin{aligned} \dim(\Delta) \leq \quad & \dim(F_k(Y_{/\Pi})) = \dim(F_l(X)) + \phi(l+1, d-1, k) + n - l - 1 \\ = \quad & \dim(F_l(X)) + (k+1)(l-k-1) - \tbinom{k+d-1}{d-1} + n - l - 1 =: U \end{aligned}$$

Step 3. We now simplify the inequality $L \leq \dim(\Delta) \leq U$ from steps 1 and 2:

$$\begin{aligned} k(l-n+1) \quad &+ \binom{k+d-1}{k-1} + \dim(F_k(X)) + \dim(F_l(X)) \\ &\leq \dim(F_l(X)) + (k+1)(l-k-1) - \binom{k+d-1}{k} + n - l - 1. \end{aligned}$$

Cancelling $F_l(X)$ and rearranging, we get

$$\begin{aligned} \dim(F_k(X)) \leq \quad & (k+1)(l-k+1) - k(l-n+1) + n - l - 1 \\ & -\tbinom{k+d-1}{k-1} - \tbinom{k+d-1}{k} \\ = \quad & (k+1)(l-k-1-(l-n+1)) - \tbinom{k+d}{d} \\ = \quad & (k+1)(n-k) - \tbinom{k+d}{d}. \end{aligned}$$

This establishes the desired upper bound $\dim(F_k(X)) \leq \phi$, which is equal to the lower bound from Chapter 1, yielding the desired equality. □

Remark 2.19. Since this last upper bound on $\dim(F_k(X))$ is sharp, the equivalent inequality $L \leq U$ of the bounds on $\dim(\Delta)$ must also be an equality, i. e. $L = \dim(\Delta) = U$. So the sharpness of our upper bound on $\dim(F_k(X)$ reflects the fact that the arguments of steps 1 and 2 were each an accurate computation of the dimension of Δ.

Also note that since X in the proof is swept out by k-planes, Theorem 1.6(c) is trivial, as argued prior to Remark 1.9. So the full result of [1] is not required for Theorem 2.13.

3. Unirationality of Smooth Hypersurfaces of Low Degree.

In this chapter we will demonstrate an important consequence of Theorem 2.13: the unirationality of smooth, low-degree hypersurfaces. Our main reference will again be [2]. In Section 3.3, however, we arrive at similar constructions to those of [2] much more straightforwardly. The methods of this chapter were originally developed for the case of a general hypersurfaces. The conclusion of the last chapter will be used to prove Corollary 3.12: we are able to bound the dimension of the variety of planes contained in the smooth fibers in order to "inflate" the image, showing surjectivity.

Note: All fibers of morphisms will now be "scheme-theoretic," as will all inclusions. A "point," however, will still refer always to a closed point.

Example 3.1. Our method of proof in this chapter generalizes that of Proposition 2.3 (a review of this example is recommended). We begin with a heuristic attempt to prove the case of quartic hypersurfaces, identifying the main obstructions to its success.

Let $X \subset \mathbb{P}^n$ be a smooth quartic hypersurface. Hoping to extend the proof of Proposition 2.3, we take n sufficiently large that we can choose an l-plane $\Gamma \subset X$ and consider the $(l+1)$-planes Θ that contain it. A general such plane Θ intersects X in the union of a residual cubic hypersurface $Y_\Theta \subset \Theta$ and the plane Γ. As before, the blow-up

$$\pi_\Gamma : \tilde{X} = \mathrm{Bl}_\Gamma(X) \longrightarrow \mathbb{P}^{n-l-1} = \Pi$$

has general fiber a cubic hypersurface $Y_\Theta \subset \mathbb{P}^{l+1}$.

Now, we expect that a general fiber Y_Θ of $\tilde{X} \to \Pi$ will be smooth.[7] In this case, Y_Θ will be unirational, by Proposition 2.3. However, if we hope to use this fact, we must be able to choose a k-plane Ω_Θ on each cubic Y_Θ. As before, we can narrow down the choice by looking only within the l-plane Γ, which meets Y_Θ in the secondary residual scheme $Z_\Theta = \Gamma \cap Y_\Theta$. As in the case of cubics, we do not expect to choose a unique such plane for each Z_Θ, i. e. a rational section. Instead we set up the incidence correspondence

$$\Psi := \{(\Theta, \Omega) \mid \Omega \subset Z_\Theta = Y_\Theta \cap \Gamma\} \subset \mathbb{P}^{n-l-1} \times \mathbb{G}(k, \Gamma).$$

(For cubics, we had $k = 0$ so that $\mathbb{G}(0, \Gamma) = \Gamma$.) As before, we form the pullback of $\tilde{X} \to \Pi = \mathbb{P}^{n-l-1}$ to this variety $\Psi \to \Pi$,

$$H := \tilde{X} \times_\Pi \Psi.$$

The family $H_1 \to \Psi$ is then a k-planed family of cubic hypersurfaces. This gives us a unirational parametrization of each cubic Y_Θ. And we should be able to further pull back H_1 to obtain a "pointed, k-planed family of hypersurfaces" H_0, dominated by a variety which will be rational—provided that Ψ itself is rational. (See Section 3.4.1 for a better description.)

The question of whether Ψ is rational can be approached as before, by considering the projection map to the second factor $\mathbb{G}(k, \Gamma)$. As in Proposition 2.9, the secondary residual hypersurfaces $\{Z_\Theta\}$ form a linear system, and each point $\Omega \subset \Gamma$ imposes a certain number of linear conditions on $\Theta \in \Pi$. Thus we may hope that Ψ is a projective bundle over the rational base $\mathbb{G}(k, \Gamma)$, and will hence be rational.

[7] In fact it will be sufficient that the general fiber have at worst isolated singularieties when $n \gg d$, as in the proof of Theorem 2.13.

There are two main obstructions to these last statements: first, the variety Ψ may not dominate $\mathbb{G}(k,\Gamma)$—then its image might not be rational, and neither might Ψ. However, we should be able to fix this by choosing $n \gg l$, so that a k-plane Ω imposes fewer than $n-l-1$ conditions on the parameters $\Theta \in \Pi$ of the linear system $\{Z_\Theta\}$—then, each Ω is contained in some hypersurface Z_Θ. Also note that although each fiber of $\Psi \to \mathbb{G}(k,\Gamma)$ is a linear space, they may not all have the same dimension—nor do all k-planes necessarily impose the same number of conditions on the series $\{Z_\Theta\}$. Thus the variety Ψ may be reducible, which is likewise true of H. However, Corollary 3.12(a) will assert that for n appropriately large, a general k-plane imposes independent conditions on a linear series in $\mathbb{P}^n$.

The second difficulty is that even if a component of $\Psi_0 \subset \Psi$ dominates $\mathbb{G}(k,n)$, and is therefore rational, we cannot be sure that Ψ_0 also dominates the other factor Π (and vice versa). If Ψ_0 does not dominate Π, then $H_0 = \tilde{X} \times_\Pi \Psi_0$ cannot dominate $\tilde{X}$ as required. However, this will be remedied in what follows by Corollary 3.12(b) to Theorem 2.13, which asserts that for $n \gg d$, *every* component of Ψ dominates Π.

Both of these failures in surjectivity, as well as the reducibility of Ψ, do in fact occur for incidence correspondences Ψ attached to certain linear series: examples are described in [2].

For the case of quartics, we will show that for

$$n \geq U(4) = 179124155$$

[8] (as defined in Corollary 3.8), these problems do not occur and a smooth quartic in $\mathbb{P}^n$ will be unirational.

3.1. **Preparatory results, and combs.** For the sake of completeness, we will prove two necessary basic facts.

Lemma 3.2. *Let $\Phi : X \to Y$ be a map of Noetherian K-schemes, and $y \in Y$ a closed point. Let $x \in X_y = X \times_Y Spec(\kappa(y)) = X \times_Y K$ be a closed point of the fiber over y (coming from a closed point of x), and let $d\Phi_x : T_xX \to T_yY$ be the differential. Then the sequence of K-vector spaces*

$$0 \longrightarrow T_xX_y \longrightarrow T_xX \xrightarrow{d\Phi_x} T_yY$$

associated to the morphisms $X_y \to X \to Y$ is exact.

Proof. Since the statement is local, we may set $X = \mathrm{Spec}(S)$ and $Y = \mathrm{Spec}(T)$. Let $\phi : T \to S$ be the ring homomorphism corresponding to Φ. Let $\mathfrak{m} \subset S$ be the maximal ideal corresponding to the point x, and let $\mathfrak{n} \subset T$ be the maximal ideal corresponding to y. The given $\Phi(x) = y$ means just that $\phi^{-1}(\mathfrak{m}) = \mathfrak{n}$, so (also by definition) there is an induced homomorphism $\phi_x : T_\mathfrak{n} \to S_\mathfrak{m}$ of the local rings such that $\phi_x(\mathfrak{n}_\mathfrak{n}) \subset \mathfrak{m}_\mathfrak{m}$.

The fiber product $X \times_Y \mathrm{Spec}(\kappa(y))$ is the spectrum of the ring

$$S \otimes_T K = S \otimes_T T_\mathfrak{n}/\mathfrak{n}_\mathfrak{n} = S \otimes_T T/\mathfrak{n}.$$

[8] As mentioned in the introduction, this number is slighly lower than it should be due to a typo in the original paper.

(The fact $T/\mathfrak{n} = T_\mathfrak{n}/\mathfrak{n}_\mathfrak{n}$ is [19] Prop. 5.1.5.) The inverse image "x" of x in the fiber X_y corresponds to the ideal $(\mathfrak{m} \otimes 1)(S \otimes_T K) = \mathfrak{m} \otimes_T K$. Furthermore, the local ring $(S \otimes_T K)_{\mathfrak{m} \otimes_T K}$ is naturally isomorphic to $S_\mathfrak{m} \otimes_T K$, under which the maximal ideal corresponds to $\mathfrak{m}_\mathfrak{m} \otimes_T K$—this is readily seen from examining the surjective ring homomorphism $S \to S \otimes_T 1$, which expresses $S \otimes_T K \approx S/(\phi(\mathfrak{n})\mathfrak{m})$, and from the exactness of localization ([16] Ch. 3).

We prove the exactness of the dual sequence to the one stated,

$$T_y^*Y \longrightarrow T_x^*X \longrightarrow T_x^*X_b \longrightarrow 0 \tag{10}$$

$$\mathfrak{n}_\mathfrak{n}/\mathfrak{n}_n^2 \xrightarrow{\phi_x} \mathfrak{m}_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}^2 \longrightarrow (\mathfrak{m}_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}^2) \otimes_T K \longrightarrow 0. \tag{11}$$

We may write out

$$(\mathfrak{m}_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}^2) \otimes_T K = (\mathfrak{m}_\mathfrak{m} \otimes_T T)/(\mathfrak{m}_\mathfrak{m}^2 \otimes_T T + \mathfrak{m}_\mathfrak{m} \otimes_T \mathfrak{n}) \approx \mathfrak{m}_\mathfrak{m}/(\mathfrak{m}_\mathfrak{m}^2 + \phi(\mathfrak{n})\mathfrak{m}_\mathfrak{m}).$$

But $\phi(\mathfrak{n})\mathfrak{m}_\mathfrak{m} = \phi_x(\mathfrak{n}_\mathfrak{n})\mathfrak{m}_\mathfrak{m}$ in the local ring $\mathscr{O}_{X,x} = S_\mathfrak{m}$ since any element $s \notin \mathfrak{n}$ is sent to $\phi(s) \notin \mathfrak{m}$. Thus the right side of the last equation is equal simply to $(\mathfrak{m}_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}^2)/(\phi_x(\mathfrak{n}_\mathfrak{n}))$, which is just the desired statement that (10) is exact. □

Proposition 3.3. *Let $X \subset B \times \mathbb{P}^n$ be a family, with X and B varieties. If the scheme-theoretic fibers X_b over closed points $b \in B$ are k-dimensional linear subspaces of $\mathbb{P}^n$, then X is a projective bundle of rank k over B, i. e. is locally trivial.*

Note. The requirement that X be a sub-bundle of a trivial bundle, i. e. $X \subset B \times \mathbb{P}^n$, is essential, as per Remark 2.2.

Proof. Given $b_0 \in B$ a closed point and $X_{b_0} \cong \mathbb{P}^k$ a fiber, choose an $(n-k-1)$-plane $\Lambda \cong \mathbb{P}^{n-k-1} \subset \mathbb{P}^n$ such that $X_{b_0} \cap \Lambda = \emptyset$. Let U be the open subset $\{b \in B \mid X_b \cap \Lambda = \emptyset\}$. Let $\pi_\Lambda : \mathbb{P}^n - \Lambda \to \mathbb{P}^k$ be the projection from Λ. Then $\mathrm{Id} \times \pi_\Lambda$ is bijective between the closed points of $X_U = \pi_1^{-1}(U)$ and $U \times \mathbb{P}^k$ (linear algebra).

From [13] Corollary 14.10, it remains to show that $\mathrm{Id} \times \pi_\Lambda$ has injective differential at all closed points $(b,p) \in X_U$. Form the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_{(b,p)}X_b & \longrightarrow & T_{(b,p)}X_U & \longrightarrow & T_bU & \longrightarrow & 0\,, \\ & & \downarrow & & \downarrow{\scriptstyle (\mathrm{Id}\times\pi_\Lambda)_{(b,p)}} & & \downarrow & & \\ 0 & \longrightarrow & T_{(b,\pi_\Lambda(p))}(\{b\} \times \mathbb{P}^k) & \longrightarrow & T_{(b,\pi_\Lambda(p))}(U \times \mathbb{P}^k) & \longrightarrow & T_bU & \longrightarrow & 0 \end{array}$$

where (if this is not obvious) the left vertical map is derived from the sequence

$$\mathbb{P}^k \cong X_b \to X_U \to U \times \mathbb{P}^k \to \mathbb{P}^k \to \{b\} \times \mathbb{P}^k,$$

which is an isomorphism since nonconstant. The left and right vertical maps in the diagram are thus isomorphisms, therefore the central map is an isomorphism.

Alternatively, notice that [14], III-56 directly implies that $X \to B$ is a flat family. So X is a fiber product with the Grassmannian bundle, hence a bundle itself. □

3.1.1. *Combs.* The following is a key object in our constructions:

Definition 3.4. A *comb* over B is a B-scheme C for which there exists a sequence

$$\begin{array}{ccccccccc} & G_1 & & G_2 & \cdots & G_{n-1} & & G_n & \\ & \cup & \searrow & \cup \quad \searrow & & & \searrow & \cup & \searrow \\ C = C_0 & \longrightarrow & C_1 & \longrightarrow C_2 & \cdots & & C_{n-1} & \longrightarrow & C_n = B, \end{array}$$

where each G_i is a Grassmann bundle over C_i, and $C_i \subset G_{i+1}$ is a dense open subset. By standard arguments, each of these maps is dominant. We will say that a map $C \to B$ for which there exists such a diagram is a *comb morphism.* Clearly, comb morphisms are closed under composition.

Proposition 3.5. *If $C \to B$ is a comb and B is rational, then C is rational.*

Proof. A Grassmann bundle over B is birational to $B \times \mathbb{G}$ for some ordinary Grassmannian $\mathbb{G}$. Grassmannians are rational by Proposition 1.1. Hence C_{n-1} is rational, and so-on. □

3.2. **Statement of results concerning unirationality.** We first must define another lower bound. Define, inductively, $L(2) = 0$, and

$$L(d) = N(d, L(d-1))$$

for $d \geq 3$.

Theorem 3.6. *Let (Γ, X, P) be an l-planed family of hypersurfaces over an integral base B, with $l \geq L(d)$. Assume Γ is contained in the smooth locus of the morphism $\pi : X \to B$. Then there exists a comb $D \to B$, and a morphism $D \to X$ such that for each closed point $b \in B$, the map $f_b : D_b \to X_b$ is dominant.*

The proof of this theorem occupies the remainder of the paper. Note that in this case, the map f will be referred to as "fiber-by-fiber" dominant. Recall that as in Remark 2.10, the smoothness requirement here is simply that fiber X_b be smooth at all points of the plane $\Gamma_b \subset X_b$.

Corollary 3.7. *If B is rational in Theorem 3.6, then X is unirational.*

Proof. From Corollary 3.5, the comb D is rational. In the theorem, $D \to X$ is dominant therefore X is unirational. □

Corollary 3.8. *For*

$$n \geq \frac{1}{L(d)+1}\binom{L(d)+d}{d} + L(d) =: U(d),$$

a smooth hypersurface $X \subset \mathbb{P}^n$ of degree d is unirational.

Proof. The requirement here is that $\phi(n, d, L(d)) \geq 0$, so that X contains an $L(d)$-plane (Ch. 1 Theorem 1.6). □

Remark 3.9. The arguments and results of this chapter, particularly Theorem 3.6, also apply to the case of a non-algebraically-closed field k of characteristic zero. However this is not true of the results of Chapter 2, and therefore Corollary 3.8 fails as stated for arbitrary k. Still, the connection between Fano varieties and unirationality persists:

Corollary 3.10. *If a smooth hypersurface $X \subset \mathbb{P}^n_k$ of degree d contains an $L(d)$-plane that is rational over k, then X is unirational over k.*

As an example of how this might work, consider the results of Section 3.1 above. Note that the $L(d)$-plane $\Lambda \subset X$ is rational over k if and only if the residue field at each closed point $x \in L(d)$ is equal to k—and in this case, the tangent spaces at x and $\Phi(x)$ are finite-dimensional k-vector-spaces, and the statement (and proof) of Lemma 3.2 makes sense.

3.3. **Constructions.** In this section we make further constructions corresponding to a given l-planed family of hypersurfaces (Γ, X, P) over B. These will formalize the techniques of Proposition 2.3 and Example 3.1. We assume that the schemes Π, Y, Z, etc., of Section 2.3.4 correspond to our given l-planed family.

Let $G := \mathrm{Grass}_k(\Gamma)$ be the family of k-planes contained in the projective bundle Γ. There is a map $\mathrm{Grass}_k(\Gamma_\Pi) \to G$ coming from the map of B-schemes

$$\mathrm{Grass}_k(\Gamma_\Pi) \xrightarrow{\sim} \mathrm{Grass}_k(\Gamma) \times_B \Pi \longrightarrow \mathrm{Grass}_k(\Gamma) = G, \tag{12}$$

from which we get another map

$$\psi : F_k(Z_{/\Pi}) \hookrightarrow \mathrm{Grass}_k(\Gamma_\Pi) \longrightarrow G.$$

Now, each fiber of $F_k(Z_{/\Pi}) \to G$ over a k-plane $\Lambda_b \in G_b$ is in fact a linear subspace of $\Pi_b \cong \mathbb{P}^{n-l-1}$. This follows form Proposition 2.9 of Section 2.4: each point of Γ_b simply imposes a linear condition on the linear series $\{Z_{\Theta_b}\}_{\Theta_b \in \Pi_b}$, so the points of Λ_b merely impose so many linear conditions on the parameters $\Theta_b \in \Pi_b$. (By applying the definition of the relative Fano variety to the discussion of Section 2.4, one is readily convinced that these linear conditions cut out the fiber in $F_k(Z_{/\Pi})$ over Λ_b scheme-theoretically, so that the fiber is a reduced linear space.)

However, we cannot be assured that all of the fibers have the same dimension; although generically, they do, by the Theorem on Fiber Dimension (1.3). Let C be the open subset of the irreducible projective variety G over which the fibers of this map ψ have the minimum possible dimension $l - \binom{k+d-1}{k}$, i. e. the set of k-planes imposing independent conditions on the linear series. At present we do not know that this is nonempty; this will follow from Corollary 3.12, if the dimension l is appropriately high. Thus, from Lemma 3.3, the restriction to C of $F_k(Z_{/\Pi}) \to G$ is in fact a projective bundle over C, which we will denote by Q :

$$F_k(Z_{/\Pi})_C =: Q \longrightarrow C \subset G.$$

We also have a morphism $Q \longrightarrow \Pi$, since Q is a sub-bundle of the restriction to C of the $\mathbb{P}^{n-l-1}$-bundle $G \times_B \Pi \longrightarrow G$.

A point of the variety Q over a closed point $b \in B$ can be thought of as a pair $(\Omega_b, \Theta_b) \in \mathrm{Grass}_k(\Gamma_b) \times \Pi_b$, such that such that $\Omega_b \subset Z_{\Theta_b}$—in other words,

$$\Omega_b \subset Z_{\Theta_b} \subset \Gamma_b \subset \Theta_b \subset P_b,$$

which also implies that

$$\Omega_b \subset Y_{\Theta_b} \subset \Theta_b$$

for a point $(\Omega_b, \Theta_b) \in Q$. If $\tilde{G} \to G$ is the universal family, then this last inclusion becomes

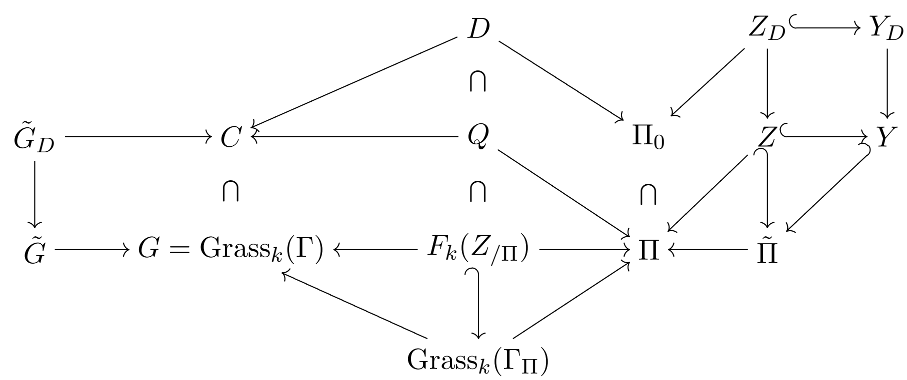
$$\tilde{G}_C \subset Q. \tag{13}$$

On the other hand, we let $\Pi_0 \subset \mathrm{DEF}(Z) \subset \Pi$ be the maximal open subset over which the fibers of $Z \to \Pi$ have the dimension $l-1$ and are smooth. In other words, $\Pi_0 \subset \mathrm{DEF}(Z) \subset \Pi$ is the set of Θ_b such that Y_{Θ_b} does not contain the entire l-plane Γ_b, and meets Γ_b in a smooth hypersurface Z_{Θ_b}.

Finally, let $D \subset Q$ be the inverse image of Π_0 in Q. Thus, the fiber over a closed point $d = (\Omega_b, \Theta_b) \in D$ of the family $Z_D \to D$ is the smooth hypersurface $Z_{\Theta_b} = Z_d$ of dimension $l-1$ in Γ_b, and $\Omega_b \subset Z_d$.

In the following diagram of B-schemes, we summarize the constructions we have made from the given family (Γ, X, P). Here, the symbol "$\subset$" indicates an open embedding, and "$\hookrightarrow$" indicates a closed embedding.

(14)



The following is the purpose of these constructions:

Proposition 3.11. *The triple $(\tilde{G}_D, Y_D, \tilde{\Pi}_D)$ is a k-planed family of hypersurfaces of degree $d-1$ over D. The bundle $\tilde{G}_D$ is contained in the smooth locus of $Y_D \to D$. Furthermore, if D and C are non-empty then*

$$D \longrightarrow C \longrightarrow B$$

is a comb over B.

Note. For this to be nontrivial, the open sets $C \subset G$ and $D \subset Q$ must be nonempty: this will be shown in Section 3.4.2 under the appropriate circumstances.

Proof. The inclusion $\tilde{G}_D \subset Y_D$ is the pullback of equation (13). The family Y_D was originally defined as a subscheme of $\tilde{\Pi}_D$ in Section 2.3.4.

For the second statement, let $d = (\Omega_b, \Theta_b) \in D$ be a closed point. Recall that $Z_d = Z_{\Theta_b}$ is by definition smooth, and $\Omega_b \subset Z_{\Theta_b} = Y_b \cap \Gamma_b$. Therefore $\Omega_b \subset (Y_d)_{sm}$, which by Remark 2.10 is equivalent to the desired statement. For the third statement,

$$D \subset Q \longrightarrow C \subset G \longrightarrow B$$

is by definition a comb morphism, provided D and C are non-empty. $\square$

3.4. **Proof of unirationality of smooth low-degree hypersurfaces.** In Section 3.4.2, we will show that for a family of hypersurfaces $X \to B$ of appropriately high dimension relative to its degree d (see section 3.2), the natural map $Y_D \to X$ is dominant. The degree of the family of hypersurfaces Y_D is equal to $d-1$, which we will use to induct on degree in Section 3.4.3.

3.4.1. *Rough description of the proof.* In the induction argument to follow, the location of the parametrizing variety itself is obscured. It is therefore instructive to give the following rough description of the proof of Theorem 3.6:

Apply Proposition 3.11 to obtain successive families of hypersurfaces $Y^{d-i}_{D_{d-i}}$ of degree $d-1, d-2, d-3...$, with each family dominating the previous ones; their bases D_{d-i} will form a comb over the original base B, provided the D_i can be chosen nonempty (we will establish this using the dimension, degree, and smoothness requirements of the theorem). This can be drawn out as follows:

$$\begin{array}{ccccccccc} X & \longleftarrow & Y^{d-1}_{D_{d-1}} & \longleftarrow & Y^{d-2}_{D_{d-2}} & \longleftarrow \cdots \longleftarrow & Y^{1}_{D_1} = D_0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ B & \longleftarrow & D_{d-1} & \longleftarrow & D_{d-2} & \longleftarrow \cdots \longleftarrow & D_1. \end{array}$$

We will arrive at a family of hypersurfaces $Y^1_{D_1}$ of degree $d = 1$ dominating all prior families $Y^{d-i}_{D_{d-i}}$. By Proposition 3.3, this last family $Y^1_{D_1}$ is then itself a projective bundle over D_1, which we may call D_0. Hence $D_0 \to D_1 \to \ldots \to B$ will be a comb over B (if non-empty), with a dominant map $D_0 \to X$ given by the composition of dominant maps $D_0 \to Y^2_{D_2} \to \cdots \to X$.

To obtain Corollary 3.7, simply observe that if the original base B is rational, then the bases D_i at each step are rational (see Proposition 3.5). This includes the last family D_0, which is rational and will dominate all the previous varieties $Y^i_{D_i}$ and X. Hence X will be unirational.

In case $B = \operatorname{Spec}(K)$ and X is a single hypersurface, the parametrizing variety is a certain subvariety of a cross-product of flag manifolds in $\mathbb{P}^r$ (see [2] for an explicit description).

3.4.2. *Dominance of $Y_D \to X$.* We proceed to establish the remaining facts which are necessary to show that $Y_D \to X$ is dominant. These are obtained directly from Theorem 2.13 of the previous chapter.

Corollary 3.12. *Let d and k be positive integers, and let $N = N(d,k)$ as defined in Section 2.5. Let $l \geq N(d,k)$ be an integer, and let*

$$\mathscr{D} = \{D_\alpha \subset \mathbb{P}^l\}_{\alpha \in \mathbb{P}^m}$$

be a base-point-free linear series of hypersurfaces of degree d in $\mathbb{P}^l$ parametrized by $\mathbb{P}^m$ (necessarily, $m \geq l$). Let

$$\Psi = \{(\alpha, \Lambda) \in \mathbb{P}^m \times \mathbb{G}(k,l) \mid \Lambda \subset D_\alpha\}$$

be the incidence correspondence. Then,

(a) *A general k-plane $\Omega \subset \mathbb{P}^l$ imposes independent conditions on the linear series $\mathscr{D}$, that is,*

$$\dim(\{\alpha \in \mathbb{P}^m \mid \Omega \subset D_\alpha\}) = m - \binom{k+d}{d} \geq 0.$$

(b) *The variety Ψ has dimension*

$$\dim(\Psi) = m + \phi(n, d, k),$$

and every irreducible component of Ψ projects surjectively onto the first factor $\mathbb{P}^m$. There is a unique irreducible component of Ψ dominating $\mathbb{G}(k, l)$, and this in particular projects surjectively onto $\mathbb{P}^m$.

Proof. This is derived by combining Lemma 2.17 and Theorem 2.13.

(a) Note that $m \geq l \geq N(d, k) \geq \binom{k+d}{d}$, which is the maximum possible number of conditions imposed by a k-plane. So $\pi_2 : \Psi \to \mathbb{G}(k, l)$ is surjective, with general fiber of dimension

$$\dim(\Psi) - \dim(\mathbb{G}(k, l)) = m + \phi(l, d, k) - (k + 1)(l - k) = m - \binom{k+d}{d}.$$

(b) Consider the projection map $\pi_1 : \Psi \to \mathbb{P}^m$. From Lemma 2.17, we know that a general member D_α of the linear series is smooth; so from Theorem 2.13, the general fiber of π_1 has dimension $\dim(F_k(D_\alpha)) = \phi(l, d, k) \geq 0$, and in particular π_1 is surjective (i. e. all hypersurfaces D_α contain k-planes. As in the proof of Theorem 2.17, set

$$S_h^o = \{\alpha \in \mathbb{P}^m \mid \dim((D_\alpha)_{\text{sing}})) = h\}.$$

Then for each $h = 0, \ldots, l - 1$, we have

$$\dim(S_h^o) \leq m - h - 1.$$

So by Corollary 2.14(b), for any $\alpha \in S_h^o$,

$$\begin{aligned} \dim(\pi_1^{-1}(\alpha)) &< \phi(l, d, k) + h + 1 \\ &\leq \phi(l, d, k) + \operatorname{codim}(S_h^o \subset \mathbb{P}^m), \end{aligned}$$

from which we may write

$$\begin{aligned} \dim(\pi_1^{-1}(S_h^o)) &\leq \dim(S_h^o) + \dim(\pi^{-1}(\Lambda)) \\ &< m + \phi(l, d, k). \end{aligned}$$

On the other hand, Ψ is cut out by $\binom{d+k}{k}$ conditions, so for each component Ψ_0 of Ψ,

$$\dim(\Psi_0) \geq m + \dim(\mathbb{G}(k, l)) - \binom{d+k}{k} = m + \phi(l, d, k).$$

Thus $\pi_1(\Psi_0) \not\subset S_h^o$ for any h, and the minimum fiber dimension over $\pi_1(\Psi_0)$ is ϕ. Corollary 1.4(b) therefore implies that $\dim(\Psi_0) = m + \phi - \phi = m$ and ψ_0 surjects onto $\mathbb{P}^m$, as desired. (Here we have "inflated" the image of Ψ_0 by bounding the dimension of the fibers, which is the contribution of Theorem 2.13.)

We have from (a) that Ψ surjects onto $\mathbb{G}(k, l)$. If two components Ψ_1, Ψ_2 both dominated, then the open sets $\Psi_1 - \Psi_2$ and $\Psi_2 - \Psi_1$ also dominate; so the generic fiber would be reducible. But each set-theoretic fiber over $\mathbb{G}(k,l)$ is a projective space, so a unique component of Ψ dominates. □

Now, assume given an l-planed family of hypersurfaces (Γ, X, P) over an integral base B, and assume that the constructions of section 3.3 have been made for this family.

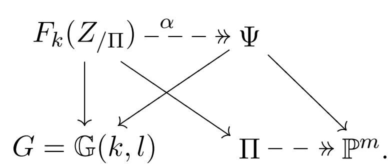
Proposition 3.13. *If $X \to B$ is smooth along Γ and $l \geq N(d-1,k)$, then D is nonempty, hence a comb over B, and the natural B-morphism $Y_D \to X$ is fiber-by-fiber dominant over B (and therefore dominant).*

Proof. (It may be helpful here to refer to the diagram (14).) As before, it will suffice to work over a single closed point $b \in B$, so we assume temporarily that $B = \mathrm{Spec}(K)$. Thus $X = X_b$ is smooth along $\Gamma = \Gamma_b \cong \mathbb{P}^l$, and by Proposition 2.9(b), the secondary residual schemes $\{Z_\Theta\}$ form a *base-point free* linear series of degree $d-1$ in Γ (a basepoint would be singular).

There is a trivial technicality coming from Proposition 2.9: the linear series $\{Z_\Theta\}$ is parametrized not by $\Pi = \mathbb{P}^{r-l-1}$ but by a quotient of Π, which we will call $\mathbb{P}^m$. There is thus a surjective map $\Pi \supset \mathrm{DEF}(Z) \twoheadrightarrow \mathbb{P}^m$ whose fibers are projective subspaces $\mathbb{P}^p \subset \mathrm{DEF}(Z)$ of rank $p = r-l-m-1$. This map induces a surjective rational map α from the Fano variety (over b),

$$F_k(Z_{/\Pi}) \subset (\mathrm{Grass}_k(\Gamma) \times_B \Pi)_b = \mathbb{G}(k,l) \times \mathbb{P}^{n-l-1},$$

onto the the incidence correspondence "Ψ"$\subset \mathbb{G}(k,l) \times \mathbb{P}^m$ of Corollary 3.12:



Each fiber of this map α is of course also a projective space $\mathbb{P}^p$.

Now, by Corollary 3.12(a), since $l \geq N(d-1,k) \geq \binom{k+d-1}{k}$, the general k-plane imposes independent conditions on the linear series $\{Z_\Theta\}$, and each fiber is non-empty. Thus $\Psi \to G$ is surjective, as is $F_k(Z_{/\Pi}) \to G$, and the open subset $C \subset G$ is non-empty. So, the projective bundle $Q \subset F_k(Z_{/\Pi})$ is a non-empty open subset. On the other hand, the open subset $\Pi_0 \subset \Pi$ is also non-empty (over b): for, $\mathrm{DEF}(Z)_b$ is non-empty by Corollary 3.12(b), and the general Z_Θ is smooth by Lemma 2.17. So $D \subset Q$, the inverse image of Π_0. Therefore, following Proposition 3.11, $D \to C \to B$ is in fact a comb.

We now show that $Y_D \to X$ dominates over b. By Corollary 3.12(b), "Ψ" dominates $\mathbb{P}^m$, hence $F_k(Z_{/\Pi})$ dominates $\Pi = \mathbb{P}^{r-l-1}$ as do the open subsets Q and D. Let $\Pi_1 \subset \Pi_0$ be an open subset contained in the image of D. Let $X_0 \subset X$ be the open subset lying outside of both Γ and the inverse image of $\Pi - \Pi_1$ under the projection $\pi_\Gamma : X - \Gamma \to \Pi$; this set X_0 is non-empty since $\Pi_1 \subset \Pi_0 \subset \mathrm{DEF}(Z) \subset \mathrm{DEF}(Y)$. Furthermore, X_0 is dense, since $X = X_b$ must be integral in order not to meet Γ in a singular point.

Let $\Theta_x = \overline{x, \Gamma}$.Choose a k-plane $\Omega_x \subset Z_{\Theta_x}$, so (Ω_x, Θ_x) is a closed point of D (possible since $l \geq N(d-1,k)$). Then because $x \notin \Gamma$, we have $x \in Y_{(\Omega_x, \Theta_x)}$. Thus x is in the image of $Y_{(\Omega_x, \Theta_x)}$, and we are done since $x \in X_0$ was general. □

3.4.3. *Proof of Theorem 3.6.* We argue by induction on the degree d. In case $d = 1$, the family of hypersurfaces is itself a comb. Now, let $d > 1$ and assume that the conclusions of the theorem hold for all degrees less than d. We must show that the theorem holds for a family (Γ, X, P) of l-planed hypersurfaces of degree d, with Γ a projective bundle of rank $l \geq L(d)$ contained in the smooth locus of $X \to B$.

Set $k = L(d-1)$, and let $(\tilde{\Omega}_D, Y_D, \tilde{\Pi}_D)$ be the k-planed family of hypersurfaces of degree $d-1$ corresponding to (Γ, X, P), from Section 3.3. By Proposition 3.11, and since $l \geq L(d) = N(d-1, L(d-1)) = N(d-1, k)$, the family $(\tilde{\Omega}_D, Y_D, \tilde{\Pi}_D)$ satisfies the hypotheses of Proposition 3.13. Therefore $D \to B$ is a comb, and $Y_D \to X$ is fiber-by-fiber dominant over B.

Also by Proposition 3.11 and the definition $k = L(d-1)$, the family $(\tilde{\Omega}_D, Y_D, \tilde{\Pi}_D)$ satisfies the induction hypothesis. Therefore we get a comb $F \to D$ and a morphism $F \to Y_D$ which is dominant fiber-by-fiber over D and hence also over B. The composition of comb morphisms

$$F \to D \to B$$

is again a comb morphism, and the composition

$$F \to Y_D \to X$$

is again a fiber-by-fiber dominant B-morphism. So F is the required comb over B dominating X fiber-by-fiber. This completes the proof of Theorem 3.6. □

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